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Second-order singular pertubative theory for gravitational lenses

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ABSTRACT

The extension of the singular perturbative approach to the second order is presented in this paper. The general expansion to the second order is derived. The second-order expansion is considered as a small correction to the first-order expansion. Using this approach, it is demonstrated that in practice the second-order expansion is reducible to a first order expansion via a re-definition of the first-order pertubative fields. Even if in usual applications the second-order correction is small the reducibility of the second-order expansion to the first-order expansion indicates a potential degeneracy issue. In general, this degeneracy is hard to break. A useful and simple second-order approximation is the thin source approximation, which offers a direct estimation of the correction. The practical application of the corrections derived in this paper is illustrated by using an elliptical NFW lens model. The second-order pertubative expansion provides a noticeable improvement, even for the simplest case of thin source approximation. To conclude, it is clear that for accurate modelization of gravitational lenses using the perturbative method the second-order perturbative expansion should be considered. In particular, an evaluation of the degeneracy due to the second-order term should be performed, for which the thin source approximation is particularly useful.

Key words: gravitational lensing: strong.

1 INTRODUCTION

The singular perturbative method is a non-parametric approach to gravitational lenses offering a direct relation between the description of the lens and the observations. The direct relation between the lens and the data minimize the degeneracy problems generally encountered in gravitational lenses modelling (see for instance Saha & Williams 2006, Wucknitz 2002, and Chiba & Takahashi 2002).

It is interesting to note that this method is general and does not depend on a particular geometry. Due to the direct relation between the equations in this method and the observations, the modelization of the lens is straightforward and free of assumptions. A direct comparison between the perturbative method and conventional method (see Alard 2010) demonstrates an un-biased and more accurate reconstruction of the lens. The method has also the potential to reconstruct very complicated lens systems, which are very difficult to model using conventional methods (see Alard 2009 and Alard 2017).

1.1 Basics of the first-order perturbative expansion

The first-order singular perturbative method was introduced a series of papers, see Alard (2007) for the basics of the method and Alard (2008), Alard (2009), Alard (2010), and Alard (2017). Let's first recall the basics of the first-order method. The main idea is to consider that gravitational arcs are a small perturbation of the perfect ring situation. In general, a larger perturbation of the circular potential will not produce very elongated images looking like gravitational arcs. A direct illustration of this fact is to consider an elliptical potential, as the ellipticity of the potential increases the angular extent of the images formed near the critical lines decrease.

The perfect ring situation is obtained when a point source is at the centre of a circular potential. The images of the central point source is an infinity of points situated on a circle. The radius of this circle is the Einstein radius associated with the circular potential. For simplicity, the Einstein radius is reduced to unity by adopting a proper set of distance units. The introduction of a non-circular perturbation to the circular potential results in the breaking of the circle with the consequence that the central point has now a finite number of images in the vicinity of the circle. In practice, the source itself is not reduced to a point but has a finite size, which is of the order of the potential perturbation. Additionally, the source may not be exactly at the centre of the circular potential $\phi_0(r)$ and as a consequence has an impact parameter, which is also of the order of the potential perturbation which we call ϵ, with $\epsilon \ll 1$. Using polar coordinates (r, θ) in the lens plane, the potential reads

$$\phi(r, \theta) = \phi_0(r) + \epsilon\psi(r, \theta). \quad (1)$$

The functional $\phi_0(r)$ in equation (1) represents the mean circular potential and $\psi(r, \theta)$ represents the small anisotropic part of the potential. The lens equation relating the lens plane coordinates $\boldsymbol{r}$ to

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the source plane coordinates $\boldsymbol{r}_S$ reads

$$\boldsymbol{r}_S = \boldsymbol{r} - \nabla\phi. \tag{2}$$

The radial deviation from the circle is of the same order as the potential perturbation, thus by introducing the deviation from the circle dr we obtain, $r = 1 + \epsilon dr$. By inserting equation (1) in the lens equation and developing to the first order in ϵ, we obtain a set of equations already presented in Alard (2007)

$$\boldsymbol{r}_S = (\kappa_2\, \mathrm{d}r - f_1)\, \boldsymbol{u}_r - \frac{\mathrm{d}f_0}{\mathrm{d}\theta}\boldsymbol{u}_{\boldsymbol{\theta}} \tag{3}$$

and

$$f_1 = \left[\frac{\partial\psi}{\partial r}\right]_{r=1};\quad f_0 = \psi(1,\theta);\quad \kappa_2 = 1 - \left[\frac{\mathrm{d}^2\phi_0}{\mathrm{d}r^2}\right]_{r=1}. \tag{4}$$

We define the source impact parameter, $\boldsymbol{r}_0 = (x_0, y_0)$ and the new variable $\tilde{r}_S = r_S - r_0$. It is convenient to re-write equation (3) using the variable $\tilde{r}_S$:

$$\tilde{r}_S = \left(\kappa_2\, \mathrm{d}r - \tilde{f}_1\right) \boldsymbol{u}_r - \frac{\mathrm{d}\tilde{f}_0}{\mathrm{d}\theta}\boldsymbol{u}_{\boldsymbol{\theta}}. \tag{5}$$

With the following definition for the corrected fields:

$$\tilde{f}_i = f_i + x_0\cos(\theta) + y_0\sin(\theta),\ i = 0, 1 \tag{6}$$

The fields f_1 and $\frac{\mathrm{d}f_0}{\mathrm{d}\theta}$ have direct and simple physical meaning. To illustrate this direct relation to the observation, we will now consider a circular source. It is straightforward to solve equation (3) and obtain the images of the circular contour of the source. For a contour with radius r_0, equation (3) is of second order in dr leading to the following two solutions:

$$\kappa_2 \mathrm{d}r = \tilde{f}_1 \pm \sqrt{r_0^2 - \left[\frac{\mathrm{d}\tilde{f}_0}{\mathrm{d}\theta}\right]^2}. \tag{7}$$

Equation (7) provides a direct relation between the contours of the images and the perturbative fields $\tilde{f}_1$ and $\frac{\mathrm{d}\tilde{f}_0}{\mathrm{d}\theta}$. The field $\tilde{f}_1$ is the mean position of the image contour at each angular position θ. While the field $\frac{\mathrm{d}\tilde{f}_0}{\mathrm{d}\theta}$ is related to the angular extent of the images. The field $\frac{\mathrm{d}\tilde{f}_0}{\mathrm{d}\theta}$ is zero at the centre of the image and has precisely the value r_0 at the image edge. As a consequence the morphology of the fields $\frac{\mathrm{d}\tilde{f}_0}{\mathrm{d}\theta}$ controls the formation of the images. This direct relation between the theory and the observations is a unique feature of the perturbative model. A direct consequence is that in this circular source model given a set of image contours it is always possible to find a solution for the two perturbative fields f_1 and $\frac{\mathrm{d}f_0}{\mathrm{d}\theta}$. In practice, the circular source model may lead to highly complex and unphysical solution for the fields, and is used only to build a first guess. The general reconstruction of the field and source contours from the image contours is carried out using equation (5). The first-order reconstruction of gravitational lenses is very successful, but for the consistency of the method it is interesting to understand and estimate the effect of the second order terms. To be more specific the effect of the second-order terms can be understood by the discussion of the two following points. The first point (i) is the estimation of the amplitude of the correction due to the second-order terms on the reconstruction of fields, and the second point (ii) is about the development of an efficient procedure to reconstruct the correction to the first order contours.

2 SECOND-ORDER EXPANSION

The perturbative development of the perfect circle situation is not limited to the first order in ϵ. The expansion may be carried out to any order. The second-order expansion of the potential requires the introduction of an additional field $f_2(\theta)$. The equations for the second-order expansion and the field f_2 reads

$$\begin{cases} \phi(r,\theta) = \phi_0(r) + \epsilon\psi(r,\theta) \\ \psi(r,\theta) = f_0(\theta) + f_1(\theta)(r-1) + f_2(\theta)\frac{(r-1)^2}{2}. \\ f_2 = \left[\frac{\partial^2\psi}{\partial r^2}\right]_{r=1} \end{cases} \tag{8}$$

Inserting in equation (2) and developing to second order in ϵ

$$\tilde{r}_S = \left(\kappa_2\, \mathrm{d}r - \kappa_3\frac{\mathrm{d}r^2}{2} - \tilde{f}_1 - f_2\mathrm{d}r\right) \boldsymbol{u}_r - \left(\frac{\mathrm{d}\tilde{f}_0}{\mathrm{d}\theta} + \left(\frac{\mathrm{d}f_1}{\mathrm{d}\theta} - \frac{\mathrm{d}f_0}{\mathrm{d}\theta}\right)\mathrm{d}r\right)\boldsymbol{u}_\theta. \tag{9}$$

With the definition of the additonal parameter κ_3

$$\kappa_3 = \left[\frac{\mathrm{d}^3\phi_0}{\mathrm{d}r^3}\right]_{r=1}.$$

The order of equation (9) in dr implies that for circular source contours the equation for dr is of fourth order instead of second order for the first order expansion (Alard 2007). However we will consider a regime where the second-order displacement dr_2 is small with respect to the first-order displacement dr_1. As a consequence

$$\mathrm{d}r = \mathrm{d}r_1 + \epsilon\mathrm{d}r_2. \tag{10}$$

Where dr_1 corresponds to the first-order expansion and dr_2 is the second-order correction. By re-expanding equation (9) to second order in ϵ using equation (10) we obtain

$$\tilde{r}_S = \left(\kappa_2\mathrm{d}r - \tilde{f}_1 - \kappa_3\frac{\mathrm{d}r_1^2}{2} - f_2\mathrm{d}r_1\right) \boldsymbol{u}_r - \left(\frac{\mathrm{d}\tilde{f}_0}{\mathrm{d}\theta} + \left(\frac{\mathrm{d}f_1}{\mathrm{d}\theta} - \frac{\mathrm{d}f_0}{\mathrm{d}\theta}\right)\mathrm{d}r_1\right)\boldsymbol{u}_\theta. \tag{11}$$

It is straightforward to reduce equation (11) to the first-order expansion equation (5) by making the following substitutions:

$$\begin{cases} f_1 = f_1 + \kappa_3\frac{\mathrm{d}r_1^2}{2} + f_2\mathrm{d}r_1 \\ \frac{\mathrm{d}f_0}{\mathrm{d}\theta} = \frac{\mathrm{d}f_0}{\mathrm{d}\theta} + \left(\frac{\mathrm{d}f_1}{\mathrm{d}\theta} - \frac{\mathrm{d}f_0}{\mathrm{d}\theta}\right)\mathrm{d}r_1. \end{cases} \tag{12}$$

Note that the second-order correction to the fields presented in equation (12) can be iterated. Once the fields have been corrected new positions for the images can be estimated and used as new entries to estimate another correction for the fields. By iterating this process a full convergence to the second-order expansion is obtained. A practical and useful approximation is to consider a source with a small size. Since the typical scale of all quantities is ϵ, by definition a small source will have a typical size r_0, which has to be of second order. We will call such a source with second order typical size a 'thin source'. As a consequence $r_0 \equiv \epsilon^2 r_0$, which also means that at the first order in ϵ the radius of the source is zero. The first-order perturbative equation for a circular source (equation 7) indicates that in this case d$r_1 = \frac{\tilde{f}_1}{\kappa_2}$. Note that a thin source implies a thin arc, equation (7) indicates that the thickness of the arc is of the order of the source radius. As a result for thin arcs the second-order expansion reads

$$\tilde{r}_S = \left(\kappa_2\mathrm{d}r - \tilde{f}_1 - \kappa_3\frac{\tilde{f}_1^2}{2\kappa_2^2} - \frac{\tilde{f}_1 f_2}{\kappa_2}\right) \boldsymbol{u}_r - \left(\frac{\mathrm{d}\tilde{f}_0}{\mathrm{d}\theta} + \left(\frac{\mathrm{d}f_1}{\mathrm{d}\theta} - \frac{\mathrm{d}f_0}{\mathrm{d}\theta}\right)\frac{\tilde{f}_1}{\kappa_2}\right)\boldsymbol{u}_\theta. \tag{13}$$

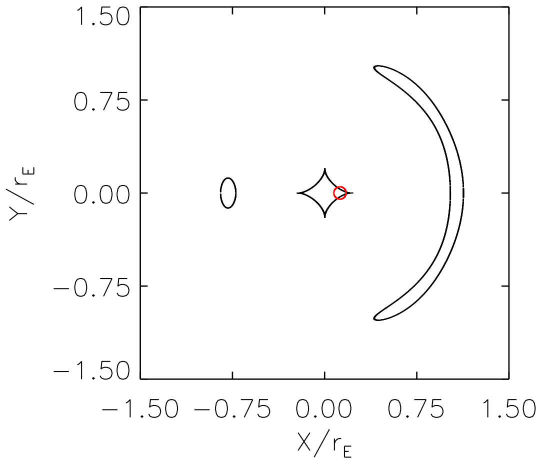

Figure 1. The source position and its associated images for the NFW lens. The images contours (black) corresponds to the circular source contour (red). The diamond shaped curve at the centre corresponds to the caustics of the NFW elliptical lens.

In the thin arc approximation, it is possible to derive an explicit substitution to recover the first order expansion:

$$\begin{cases} f_1 = f_1 + \kappa_3 \frac{\tilde{f}_1^2}{2\kappa_2^2} + f_2 \frac{\tilde{f}_1}{\kappa_2} \\ \frac{\mathrm{d}f_0}{\mathrm{d}\theta} = \frac{\mathrm{d}f_0}{\mathrm{d}\theta} + \left(\frac{\mathrm{d}f_1}{\mathrm{d}\theta} - \frac{\mathrm{d}f_0}{\mathrm{d}\theta}\right) \frac{\tilde{f}_1}{\kappa_2}. \end{cases} \tag{14}$$

The corrective terms in equation (14) could be included in f_1 and f_0 or be considered as genuine order 2 corrections. As a consequence equation (14) describes explicitly the degeneracy in the reconstruction of the fields. Breaking this degeneracy is difficult since it would require information at sufficient radial distance for the same angular position θ, which is a requirement very hard to fulfill in practice. The best opportunity to break this degeneracy would be to have several sources situated at different distances and thus having different effective Einstein radius.

3 PRACTICAL IMPLEMENTATION BY USING A NUMERICAL EXPERIMENT

We consider the contour of a circular source situated near the caustic of a NFW halo lens. The potential for an elliptical NFW halo is (Meneghetti, Bartelmann percent Moscardini 2003)

$$\begin{cases} \phi(u) = \frac{1}{1-\ln(2)} g(u). \\ u = \sqrt{((1-\eta)x^2 + (1+\eta)y^2)}. \end{cases} \tag{15}$$

The parameter η is related to the ellipticity of the halo. The potential normalization implies that the associated Einstein radius is equal to the typical halo size, which is a common situation for gravitational lenses. The definition of the function $g(u)$ reads

$$g(u) = \frac{1}{2}\ln\left(\frac{u}{2}\right)^2 + \begin{cases} 2\mathrm{arctan}^2\left(\sqrt{\frac{u-1}{u+1}}\right) & u \geq 1 \\ -2\mathrm{arctanh}^2\left(\sqrt{\frac{1-u}{u+1}}\right) & u < 1. \end{cases} \tag{16}$$

The source configuration in the potential defined in equation (15) is presented in Fig. 1 with the images of the source circular contour. All reconstructions of the circular source contour with radius r_0 are performed using the first-order formula(Alard 2007) and the modified fields defined in equations (12) and (14) for the second-order reconstructions. The first-order circular contour equation is

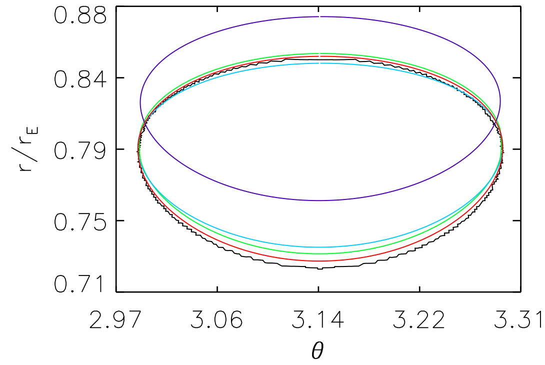

Figure 2. A detailed view of the left-hand side of the image (see Fig.1). The actual image contour (black) is superimposed with the first-order reconstruction (blue), the second-order thin source approximation (light blue), the first iteration of the second-order reconstruction (green) and the iterated second order reconstruction (red).

$$\kappa_2 \mathrm{d}r = f_1 \pm \sqrt{r_0^2 - \left[\frac{\mathrm{d}f_0}{\mathrm{d}\theta}\right]^2}. \tag{17}$$

The results obtained in Fig. 2 indicates that the first-order reconstruction is not very accurate for the left-hand side of the image. All second-order expansions provide a clear improvement in accuracy. Even the simplest second-order expansion, the thin source approximation (see equations 13 and 14) already represents a significant improvement over the first-order expansion. The first iteration of the second-order expansion (see equations 11 and 12) is more accurate than the thin source approximation. Iterating the second order allows us to reach the level of accuracy corresponding precisely to the second-order perturbative expansion. It is interesting to note that the typical order of the second-order correction is the same as the error due to the noise in the data for reconstruction performed using high-quality *HST* data. A good illustration can be found in Alard (2017), where the typical amplitude for both the second-order correction and the error due to the noise is about 10^{-3} in units of the Einstein radius (see Section 2.5 and 3.1). Even if the second-order correction is compared to the error due to the noise, this correction is not an error in itself but a small degeneracy. The presence of this degeneracy issue does not alter the accuracy of the modelling, which still reach a level that is close the statistical expectation (see Alard 2017, table 1). The results for the right-hand side of the image (see Fig. 3) are similar although the first-order approximation is noticeably more accurate for this image.

4 CONCLUSION

It is relatively simple to estimate the second-order perturbative expansion as a correction of the first-order expansion. In particular, the correction in the thin source limit is straightforward and provide a noticeable improvement over the first-order pertubative expansion. The iterative full second-order correction converge to the second-order perturbative correction but in most cases provides only a small additional improvement with respect to the thin source approximation. Additionally, it is interesting to note that larger sources can always be de-composed in a number of thinner sources for which the thin source approximation is valid. Another important issue is the problem of the degeneracy of the second-order correction. Even if in most case the correction is small, the problem of the possible degeneracy of first-order expansion should be addressed.

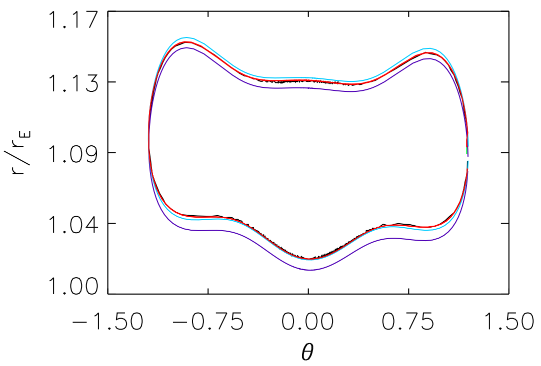

Figure 3. A detailed view of the right-hand side of the image (see Fig. 1). The actual image contour (black) is superimposed with the first-order reconstruction (blue), the second-order thin source approximation (light blue), the first iteration of the second-order reconstruction (green) and the iterated second-order reconstruction (red).

For an evaluation of the amplitude of the degenerate term, the thin source approximation should be particularly useful as it offers a direct estimation. In some particular application when the degeneracy of the second-order term can be broken, the full estimation of the second-order expansion should be useful.

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