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Influence of tides in viscoelastic bodies of planet and satellite on the satellite's orbital motion

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ABSTRACT

The problem of influence of tidal friction in both planetary and satellite bodies upon satellite's orbital motion is considered. Using the differential equations in satellite's rectangular planetocentric coordinates, the differential equations describing the changes in semimajor axis and eccentricity are derived. The equations in rectangular coordinates were taken from earlier works on the problem. The calculations carried out for a number of test examples prove that the averaged solutions of equations in coordinates and precise solutions of averaged equations in the Keplerian elements are identical. For the problem of tides raised on planet's body, it was found that, if satellite's mean motion n is equal to $\frac{11}{18}\Omega$, where Ω is the planet's angular rotation rate, the orbital eccentricity does not change. This conclusion is in agreement with the results of other authors. It was also found that there is essential discrepancy between the equations in the elements obtained in this paper and analogous equations published by earlier researchers.

Key words: Planets and satellites: general.

1 INTRODUCTION

In the last years, the problem of influence of tides raised on viscoelastic bodies of planets and satellites upon satellite orbital motion has become a topical issue. The accuracy of observations of the major satellites of Jupiter, Saturn, Uranus, and Neptune has greatly increased. In addition, in course of time, the intervals of observations have naturally become larger. These factors gave an impetus to attempts to determine from observations those physical parameters of planets and satellites that define the forces of tidal friction. The tidal bulge moves in body's interior creating a torque acting upon satellite. The force is proportional to the ratio k_2/Q, where k_2 is the Love number that characterizes the deformability of a body, Q is the quality factor characterizing the viscosity of body's interior. It follows from the equations of motion that observations do not allow us to obtain independent values of k_2 and Q but only their ratio k_2/Q.

Lainey et al. (2009a), taking all available at the time astrometric observations of the Galilean satellites of Jupiter, determined the ratio k_2/Q for both Jupiter and its satellite Io. In the same way, using astrometric observations, Lainey et al. (2012) obtained new values of tidal dissipation ratio of Saturn that turned out to be 10 times greater than the value obtained from theoretical considerations. Moreover, an unexpectedly high value of Mimas' secular acceleration caused by the tides on the satellite's body was obtained.

When determining the parameters from observations, the usual practice is to carry out numerical integration of the equations of motion in rectangular coordinates. Hence, it is necessary to have expression for perturbing acceleration caused by tidal forces. Such equations have been deduced by earlier researchers (see references below).

The orbital evolution of a satellite caused by tidal forces is better to study by considering the changes in two key parameters: semimajor axis a and eccentricity e. It is the changes of these two parameters that determine satellite's fate, that is whether it falls down to planet or moves away from it. To this end, differential equations for these elements have been derived in a number of papers. Neglecting small short-period perturbations, two equations are usually obtained which in general form look like these:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{k_2}{Q} A_a(a, e), \quad \frac{\mathrm{d}e}{\mathrm{d}t} = \frac{k_2}{Q} A_e(a, e).$$

Separate equations of this kind are composed for both the problem of taking into account the tides on planet's body and the problem for the tides on satellite. Both problems give differing equations, but it is possible to take into account both effects in one system of equations.

For the problem in consideration, differential equations in the Keplerian elements have been published in earlier papers. In particular, they can be found in Lainey et al. (2012). In the paper (Lainey et al. 2012; equation 2), the equation of motion of a tidally perturbed body is written down in a form corresponding to the tidal model of

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Mignard (1979, 1980) and Hut (1981). Lainey et al. (2012; equations A1 and A2) also provide the expressions for da/dt and de/dt, both for a non-synchronized and a synchronized body, altogether four expressions. Below we shall demonstrate that two of those four expressions are in disagreement with the Mignard–Hut tidal model (in fact, one of those expressions is in error, while the other follows from the general Kaula expansion, provided k_2/Q in that expansion is set frequency-independent, an assumption different from the Mignard–Hut model). Providing those expressions, Lainey et al. (2012; equations A1 and A2) refer to the work by Peale & Cassen (1978), which deals with tidal heating and contains no expressions for the tidal evolution of orbital elements.

In order to improve methodology, to clarify the possibility of determining the parameters of tidal friction from observations as well as to study orbital evolution, it would be interesting to compare solutions of the equations in coordinates with solutions of the equations in Keplerian elements. This is the aim that was set in this paper.

2 BASIC FORMULAE OBTAINED BY EARLIER AUTHORS

Let us consider differential equations of satellite motion in rectangular planetocentric coordinates published in earlier works.

The equations necessary to solve the problem have been derived by Mignard (1979) who studied the influence of tides in viscoelastic interior of the Earth upon motion of the Moon. We take the formula (5) of this paper. Later the theory has been developed in Mignard (1980).

Tidal evolution in close binary systems was studied in the paper Hut (1981). The leading contribution to the tidal perturbing force is expressed by Hut (1981) with his formula (8) which is in agreement with the result of Mignard (1979).

Lainey, Dehant & Patzold (2007) obtained the solution of the problem of influence of tides on the body of Mars upon the motion of Phobos. The authors used the equations of satellite motion in rectangular coordinates taken from Mignard (1980). The corresponding formula in Lainey et al. (2007) has the reference (3).

Lainey et al. (2009a) extended the formulae for perturbing acceleration caused by the tides in planet's body to the case of tides in viscoelastic satellite body influencing upon its orbital motion. However there were no detailed derivations of the formulae. They were just declared and given without explanations in Supporting Information section of the paper [see the formulae (1) and (2) in Lainey et al. (2009b)]. The formulae have generalized form for both tides raised on planet and for those raised on satellite. Later the same equations were also published in Lainey et al. [2012; see the formulae (1) and (2)].

It is necessary to note that explanations to the formulae (1) and (2) in Lainey et al. (2012) have errors. The phrase '... and F^T_{lk} is the force received by P_l from the tides it raises on P_k' should be read as: '... and F^T_{lk} is the force received by P_k from the tides it raises on P_l'.

If we read the phrase as it is written, then we come to the conclusion that the formula does not correspond to its particular case considered by Mignard (1979) and Lainey et al. (2007). Actually, taking in equation (1) in Lainey et al. (2012) the term corresponding to the tides raised on planet's body (index 0) by satellite (index i), that is the term

$$\frac{m_0+m_i}{m_0 m_i}F^T_{i0} = -\frac{3k_2 G m_i^2 R^5 \Delta t}{r_{0i}}\left\{\frac{2\mathbf{r_{0i}}(\mathbf{r_{0i}}\mathbf{v_{0i}})}{r_{0i}^8} + [\mathbf{r_{0i}}\times\Omega] + \mathbf{v_{0i}}\right\},$$

and taking into account that (in notations of the paper in consideration) $\mathbf{r_{0i}} = \mathbf{r_0} - \mathbf{r_i} = -\mathbf{r}$, $\mathbf{v_{0i}} = \mathbf{v_0} - \mathbf{v_i} = -\mathbf{v}$, where $\mathbf{r}$ and $\mathbf{v}$ are planetocentric vectors of satellite's positions and velocities, we arrive at the following erroneous expression

$$\frac{m_0+m_i}{m_0 m_i}F^T_{i0} = \frac{3k_2 G m_i^2 R^5 \Delta t}{r}\left\{\frac{2\mathbf{r}(\mathbf{r}\mathbf{v})}{r^8} + [\mathbf{r}\times\Omega] + \mathbf{v}\right\}.$$

In contrast to the corresponding term in equation (5) in Mignard (1979) as well as in equation (3) in Lainey et al. (2007), this expression has an opposite sign.

Let us use the generalized form of the equations of motion of planetary satellite as given in Lainey et al. (2009b) and Lainey et al. (2012). However, we shall make some simplifications leaving only the terms that are of fundamental importance for further analysis. First, we leave only the main term corresponding to the planet's attraction as a material point and the terms describing tidal effects. Secondly, we neglect satellite's mass in comparison with that of the planet. This assumption is quite justifiable since the masses of the satellites are really small compared to the planetary masses. For the values in the formulae we shall use other notations than those in the papers mentioned earlier. As in these papers, the equations of motions we write in satellite's rectangular coordinates referred to the planetocentric reference frame. For convenience, we use equations for two separate problems: the problem of satellite motion influenced by the tides raised on planet's viscoelastic body and that where satellite's motion is perturbed by the tides raised on viscoelastic body of the satellite itself.

We use the following notations:
R_p - planet's radius,
R_s - satellite's radius,
G_M - gravitational parameter of the planet,
G_s - gravitational parameter of the satellite,
a - semimajor axis of the satellite's orbit,
n - mean motion of the satellite,
k_2 - Love number of the planet (dimensionless),
$k_2^{(s)}$ - Love number of the satellite (dimensionless),
Δt_p - time lag of tidal bulge in the planet's body,
Δt_s - time lag of tidal bulge in the satellite's body,
Q_p - quality factor of the planet,
Q_s - quality factor of the satellite,
$\mathbf{\Omega}$ - vector of the planet's rotation rate,
$\mathbf{\Omega_s}$ - vector of the satellite's rotation rate.

Note that Δt_p and Δt_s are assumed to be positive. The satellite's position and velocity are given by the vectors $\mathbf{r}$ and $\mathbf{v}$.

Referring to formulae (1) and (2) in Lainey et al. (2009b, 2012), under the assumptions made above and with adopted notations, we write the differential equations of satellite motion in the following form for the case of tides on the planet:

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -\frac{G_M}{r^3}\mathbf{r} - \frac{3k_2 G_s R_p^5}{r^8}\Delta t_p\left[\frac{2\mathbf{r}(\mathbf{r}\mathbf{v})}{r^2} + [\mathbf{r}\Omega] + \mathbf{v}\right], \quad (1)$$

and for the case of tides on the satellite:

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -\frac{G_M}{r^3}\mathbf{r} - \frac{3k_2^{(s)} G_M R_s^5}{r^8}\frac{G_M}{G_s}\Delta t_s\left[\frac{2\mathbf{r}(\mathbf{r}\mathbf{v})}{r^2} + [\mathbf{r}\Omega_s] + \mathbf{v}\right]. \quad (2)$$

Here $[\mathbf{r\Omega}]$ and $[\mathbf{r\Omega_s}]$ are vector cross products, $(\mathbf{rv})$ is vector dot product. These equations are consistent with the equation (8) from Hut (1981).

The right-hand sides of the equations (1) and (2) determine the acceleration of a satellite for given values of all the quantities entering here, where only G_M, G_s, R_p, and R_s are assumed to be constant.

To simplify further analysis, we introduce some new notations and slightly transform the equations. Let us introduce an arbitrary value $\bar{a}$ whose magnitude is taken to be equal to averaged value of satellite's semimajor axis. We use the well-known relationship between the Keplerian elements

$$n^2 a^3 = G_{\rm M}.$$

We introduce dimensionless constants $K_{\rm p}$ and $K_{\rm s}$ that are defined as follows:

$$K_{\rm p} = \frac{3R_{\rm p}^5}{\bar{a}^5}\frac{G_{\rm s}}{G_{\rm M}},$$

$$K_{\rm s} = \frac{3R_{\rm s}^5}{\bar{a}^5}\frac{G_{\rm M}}{G_{\rm s}}.$$

We use also the notations

$$K_2^{(\rm p)} = k_2 \Delta t_{\rm p}, \quad (3)$$

$$K_2^{(\rm s)} = k_2^{(\rm s)} \Delta t_{\rm s}. \quad (4)$$

Now, with the new notations the equations become as follows for the case of tides raised on planet's body:

$$\frac{{\rm d}^2\mathbf{r}}{{\rm d}t^2} = -\frac{G_{\rm M}}{r^3}\mathbf{r} - K_2^{(\rm p)} K_{\rm p} \frac{\bar{a}^5 a^3}{r^8} n^2 \left[\frac{2\mathbf{r}(\mathbf{rv})}{r^2} + [\mathbf{r}\Omega] + \mathbf{v}\right], \quad (5)$$

and for the case of tides on the satellite:

$$\frac{{\rm d}^2\mathbf{r}}{{\rm d}t^2} = -\frac{G_{\rm M}}{r^3}\mathbf{r} - K_2^{(\rm s)} K_{\rm s} \frac{\bar{a}^5 a^3}{r^8} n^2 \left[\frac{2\mathbf{r}(\mathbf{rv})}{r^2} + [\mathbf{r}\Omega_{\rm s}] + \mathbf{v}\right]. \quad (6)$$

Here the coefficients $K_2^{(\rm p)}$ and $K_2^{(\rm s)}$ defined by (3) and (4) can have any values at a given time. In the particular case $\Delta t_{\rm p}$ and $\Delta t_{\rm s}$ can be constant.

It was noted in Lainey et al. (2012) that the time lags $\Delta t_{\rm p}$ and $\Delta t_{\rm s}$ depend on the so-called tidal frequency, i.e. the velocity of the tidal wave run on the surface. That is why they cannot be taken as constants. Therefore in our subsequent calculations, we consider the coefficient $K_2^{(\rm p)}$ as an arbitrary function of $|\mathbf{\Omega}| - n$ and $K_2^{(\rm s)}$ as an arbitrary function of n.

The papers mentioned above explain the relationship between the time lags of tidal bulge and quality factors. According to these explanations, we have

$$K_2^{(\rm p)} = \frac{k_2}{2Q_p(|\Omega| - n)}, \quad K_2^{(\rm s)} = \frac{k_2^{(\rm s)}}{Q_{\rm s} n}. \quad (7)$$

It is assumed that the planet is rotating faster than the satellite moves along its orbit.

Note that, instead of quality factor Q, the parameters arctan Q or arcsin Q are used in some papers. Such a change of parameters, however, is not substantial in this work.

Mathematically, the results of this paper would remain valid even if $K_2^{(\rm p)}$ were an arbitrary function of $|\mathbf{\Omega}| - n$ and $K_2^{(\rm s)}$ were an arbitrary function of n. We however concentrate in our calculations on the dependencies (7) and assume parameters $Q_{\rm p}$, $Q_{\rm s}$, k_2, and $k_2^{(\rm s)}$ to be constant.

A general theory of land tides was pioneered by Darwin (1879) and furthered by Kaula (1964). As was demonstrated by Efroimsky & Makarov (2013; Section 5), the Mignard–Hut model is a particular case of that general theory. In our paper, we shall stick to the Mignard–Hut model implemented by the above equations (1) and (2).

3 SOLVING THE EQUATIONS FOR RECTANGULAR COORDINATES

Solving the equations obtained above at sufficiently large time interval can give us the picture of evolution of satellite's orbital parameters caused by the tides raised on viscoelastic bodies of both planet and satellite. That is exactly what interests researchers in this problem. We shall try to obtain the sought properties of satellite motions. Since exact analytical solution is not possible in this case, we have to use the methods of numerical integration.

We performed numerical integration of equations (5) and (6) at a certain sufficiently large time interval and obtained planetocentric coordinates and velocities of satellite for a series of time instants with constant stepsize. For each time instant, osculating Keplerian elements were computed. What is interesting for us here is the variation of the elements in time, particularly the changes in semimajor axis a and eccentricity e.

The orbits of real major satellites of Jupiter, Saturn, Uranus, and Neptune have small inclinations to their planets' equatorial planes. Let us consider a hypothetic case close to real one when a satellite moves in invariable plane, the axes of rotation of both planet and satellite being normal to this plane. Then the vectors $[\mathbf{r\Omega}]$ and $[\mathbf{r\Omega_s}]$ lie in the plane of motion. Hence, all acting forces lie in the same plane and satellite motions occur in one plane too. That is why, in solving the equations (5) and (6), we can restrict ourselves to modelling 2D motions only.

In practical calculations, the outcome depends on the spin rate $\mathbf{\Omega_s}$ of the satellite. We shall assume that it is synchronized with the mean motion.

Synchronism of the satellite rotation in the problem in consideration was intensively studied by Rodríguez, Ferraz-Mello & Hussmann (2008), Williams & Efroimsky (2012), Makarov & Efroimsky (2013) and Makarov (2015). It was found that if the satellite is perfectly oblate and only the tidal bulge is taken into account, the stable configuration is not the 1:1 spin-orbit resonance but a so-called pseudosynchronous spin state, a situation where the rotation rate is slightly exceeding the mean motion. In this work we state that the satellite is assumed to have some permanent triaxiality preserving it from pseudosynchronism and making the exact 1:1 spin-orbit resonance possible.

In publications (Lainey et al. 2009b, 2012) the satellite is assumed synchronized.

Physical parameters were taken to be close to those of the major Uranian satellites. The following constants were adopted as planetary parameters:

$G_{\rm M} = 5793939.3\,{\rm km^3 s^{-2}}$, $\Omega = 501.1600928\,{\rm deg\,d^{-1}}$.

The coefficients in the equations are taken to be as follows:

$K_{\rm p}\frac{k_2}{Q_{\rm p}} = 0.1 \times 10^{-6}$,

$K_{\rm s}\frac{k_2^{(\rm s)}}{Q_s} = 10.0 \times 10^{-6}$.

These values do not correspond to real possible values of viscosity parameters of Uranus and its satellites. However, these exaggerated values of the coefficients allow us to see the peculiarities of the solution. The constant $\bar{a}$ was taken to be equal to the initial value of satellite's semimajor axis.

Initial conditions for solution of the differential equations of motion were taken in two sets:

1. $a = 190940.453$ km,
2. $a = 114820.064$ km.

The initial eccentricity in both cases is 0.002. The first set of orbital parameters is very close to those of the Uranian moon Ariel. The second set is considered because it demonstrates some

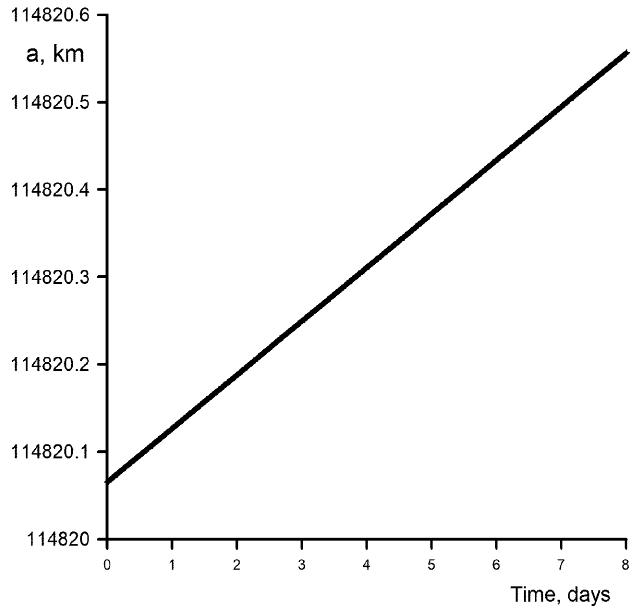

Figure 1. Changes in semimajor axis of satellite at 8-d time interval caused by tidal friction in planet's interior. The second set of initial conditions is used.

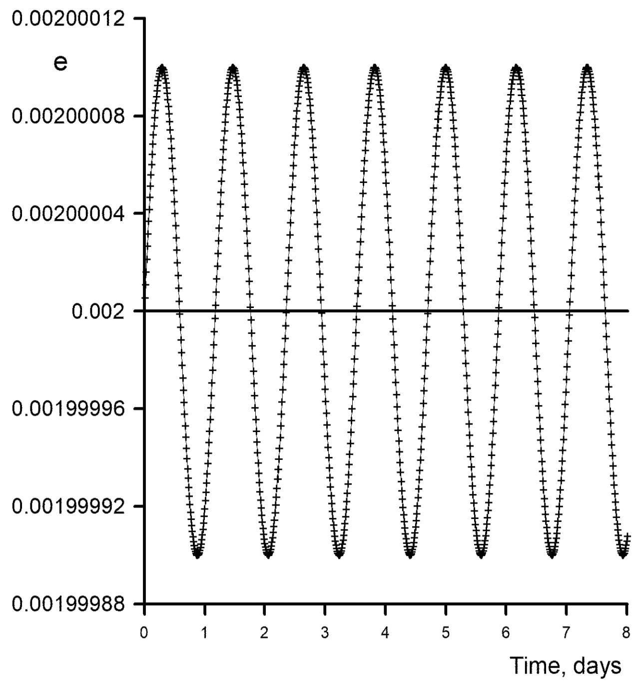

Figure 2. Changes in eccentricity of satellite at 8-d time interval caused by tidal friction in planet's interior. The second set of initial conditions is used.

peculiarities in orbital evolution (see below). In performing the numerical integration, it was supposed that the satellite's starting point is in the pericentre of its orbit.

In order to see the character and magnitude of short-period changes in osculating elements, the values of semimajor axis and eccentricity were first computed at the time interval of 8 d with the stepsize 0.01 d. It is those changes computed for the second set of initial conditions in the problem of taking into account tidal dissipation in planetary body that are shown in the Figs 1 and 2. Because

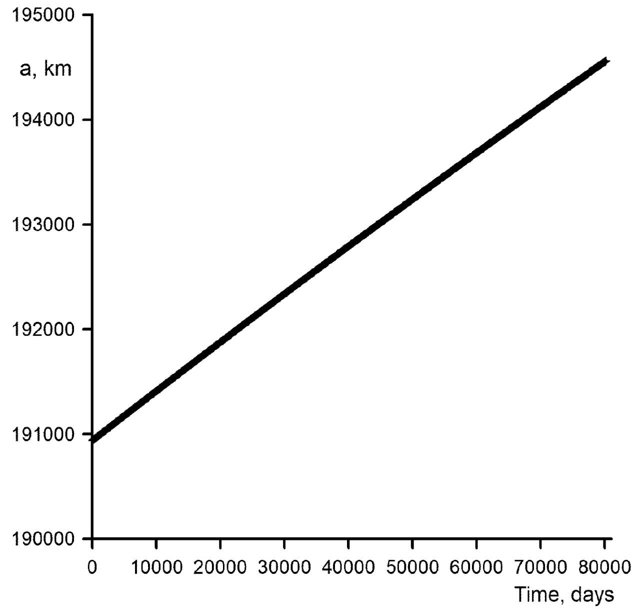

Figure 3. Change in satellite's semimajor axis at 80200-d (220-yr) time interval caused by tidal friction in planet's interior. The first set of initial conditions is used.

of strong secular perturbations, it is not possible to see short-period oscillations of semimajor axis in the plot. However, the eccentricity does manifest oscillations with the period equal to its orbital period. The plots demonstrate that short-period variations of the osculating elements *a* and *e* are extremely small and cannot characterize tidal evolution of satellite's orbit. In the same way, we obtained small amplitudes of short-period oscillations in the elements in all other cases that were considered.

In studying the orbital evolution, integration was performed at the time interval of 80 200.0 d, i.e. about 220 yr. The results were output with the stepsize 100 d (the data see below).

The Figs 3–6 show the changes in semimajor axis and eccentricity of satellite's orbit caused by tidal friction in planetary interior for both sets of initial conditions.

Note that, for the second set of initial conditions, the perturbing influence of tides raised on planet's body results in that the eccentricity is almost constant in the beginning of the time interval but increases with the growth of semimajor axis. Here the initial value of semimajor axis was especially chosen so that to demonstrate the peculiarity of solution in this case. The way this value was obtained is explained below.

Figs 7–10 show the changes in semimajor axis and eccentricity of satellite's orbit caused by tidal friction in satellite's body for both sets of initial conditions.

It is necessary to note that the changes in semimajor axes and eccentricities represented in the plots reflect exactly the real evolution of orbit due to the influence of tides in viscoelastic bodies of planet and satellite. The reliability of the results is based on the reliability of the equations of motion of satellite in rectangular coordinates that were taken from the works mentioned above.

4 TRANSITION TO THE DIFFERENTIAL EQUATIONS IN KEPLERIAN ELEMENTS

When studying planetary satellite motions at large time intervals, it is most interesting to look at the behaviour of semimajor axis *a*

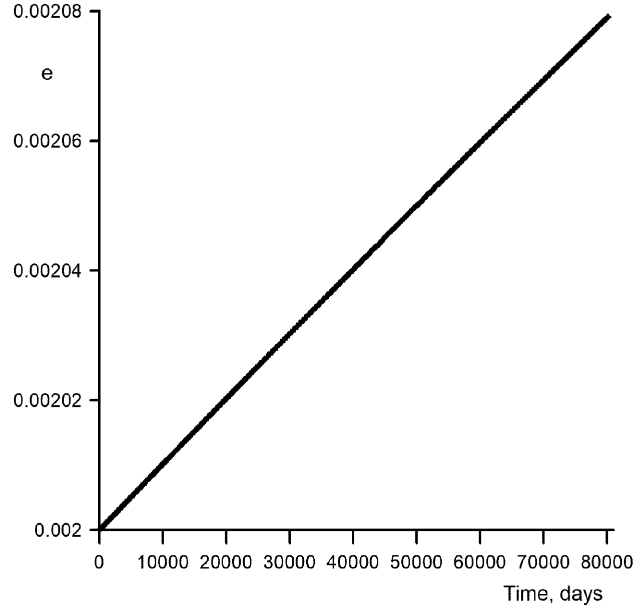

Figure 4. Change in satellite's eccentricity at 80200-d (220-yr) time interval caused by tidal friction in planet's interior. The first set of initial conditions is used.

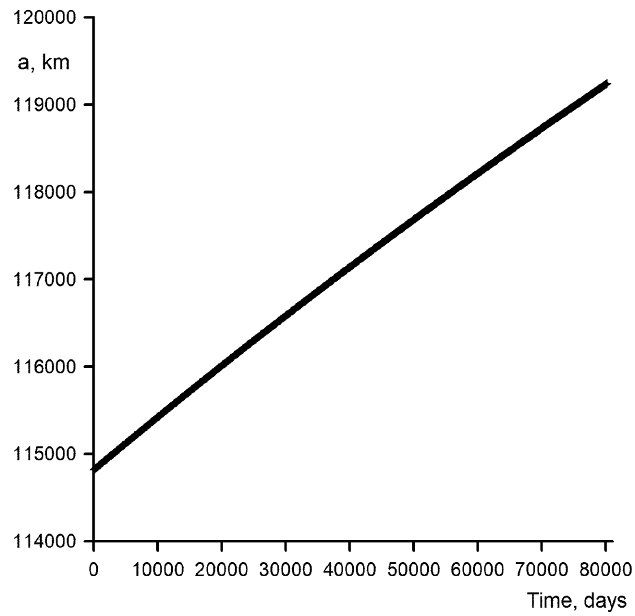

Figure 5. Change in satellite's semimajor axis at 80200-d (220-yr) time interval caused by tidal friction in planet's interior. The second set of initial conditions is used.

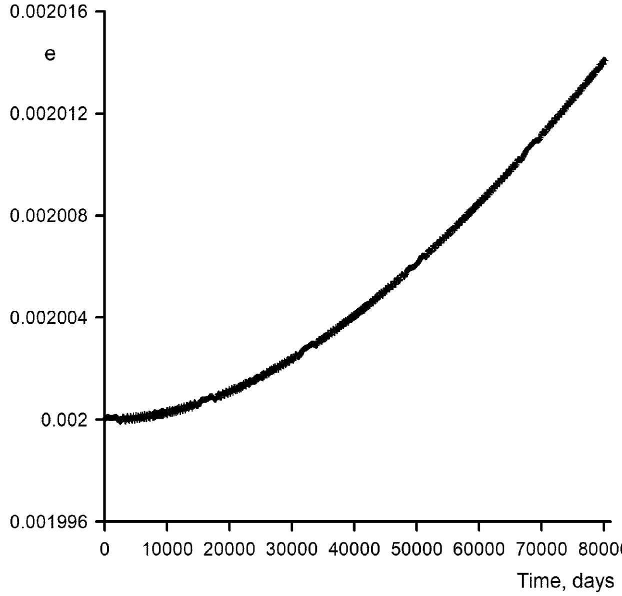

Figure 6. Change in satellite's eccentricity at 80200-d (220-yr) time interval caused by tidal friction in planet's interior. The second set of initial conditions is used.

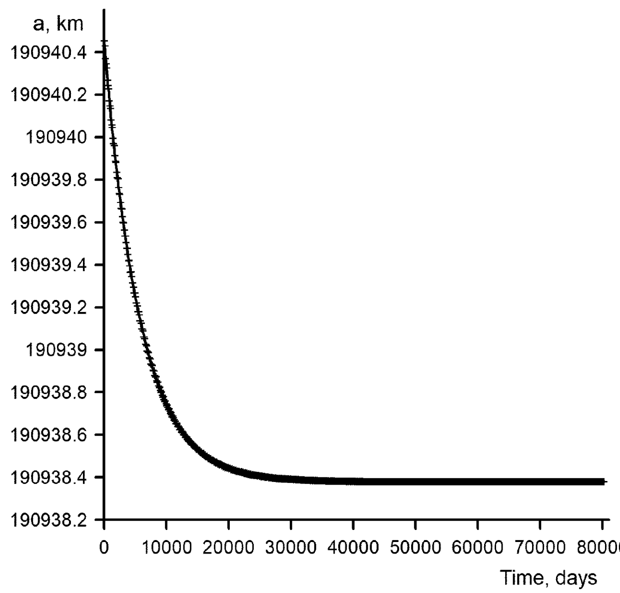

Figure 7. Change in satellite's semimajor axis at 80200-d (220-yr) time interval caused by tidal friction in satellite's interior. The first set of initial conditions is used.

and eccentricity e. It is these parameters that describe the satellite's fate. Because of tidal dissipation of mechanical energy, a and e can change in such a way that the satellite can either fall to the planet or move away from it. That is why in many works devoted to the orbital evolution differential equations for semimajor axis and eccentricity are composed. We also made an attempt to compose and solve such equations.

Since in this problem, without loss of generality, we can consider only planar motions, no inclinations or longitudes of ascending node are involved. It is also obvious that longitude of pericentre and mean anomaly at epoch do not determine the orbital evolution of satellite. It is for these reasons that we restricted ourselves to composing only equations for semimajor axis a and eccentricity e.

To derive the sought equations, we use the equations in a and e taken from Tisserand (1889), p. 433. They are as follows:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{n\sqrt{1-e^2}}\left[e\sin f\, R + \frac{a(1-e^2)}{r}\, T\right], \tag{8}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{\sqrt{1-e^2}}{na}[\sin f\, R + (\cos f + \cos E)\, T], \tag{9}$$

where f is the true anomaly, E the eccentric anomaly, R the radial component of perturbing acceleration, and T its tangential

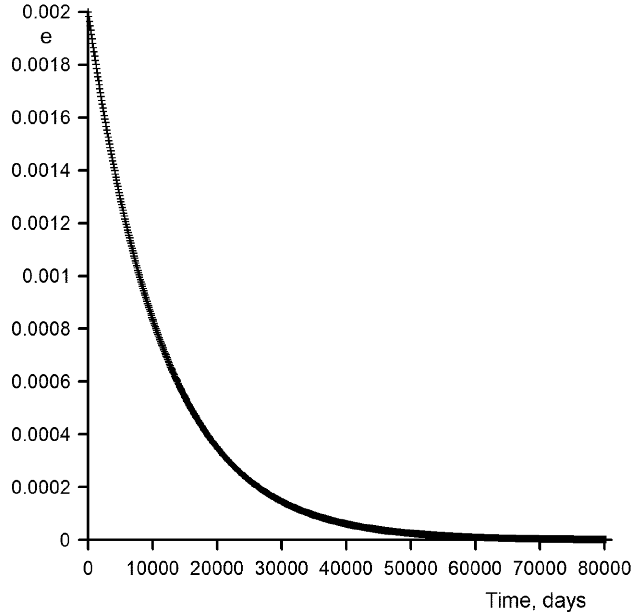

Figure 8. Change in satellite's eccentricity at 80200-d (220-yr) time interval caused by tidal friction in satellite's interior. The first set of initial conditions is used.

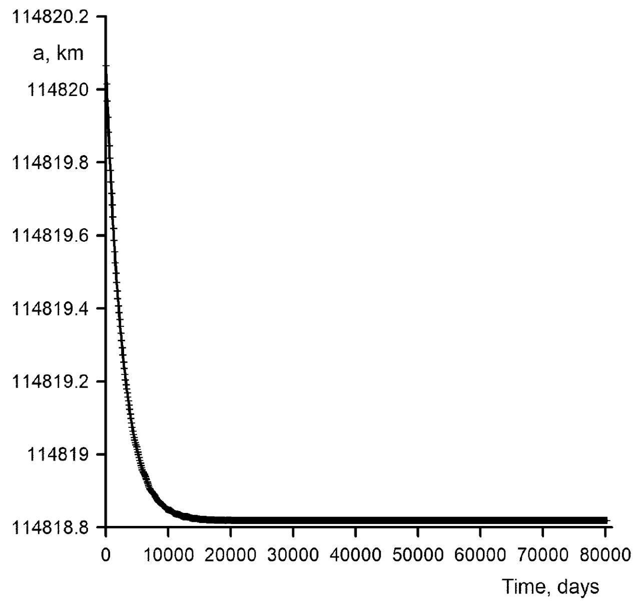

Figure 9. Change in satellite's semimajor axis at 80200-d (220-yr) time interval caused by tidal friction in satellite's interior. The second set of initial conditions is used.

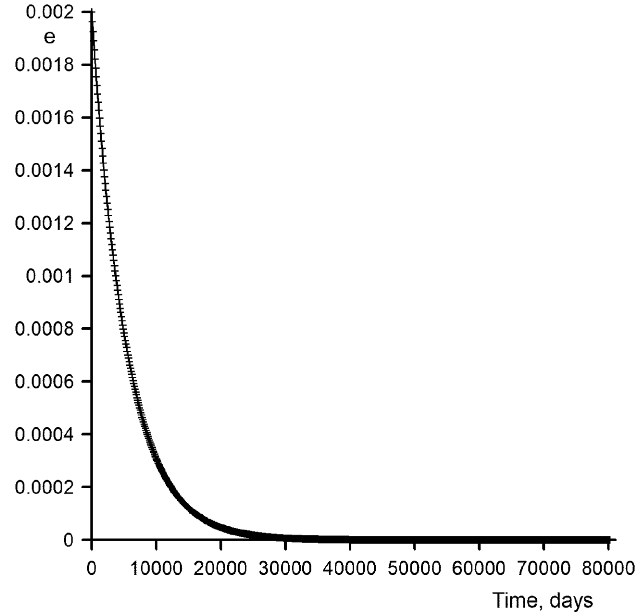

Figure 10. Change in satellite's eccentricity at 80200-d (220-yr) time interval caused by tidal friction in satellite's interior. The second set of initial conditions is used.

component. The perturbing acceleration is given by its components in the right-hand sides of (5) and (6). The sought equations will be derived separately for tides raised on planet's body and for those raised on the satellite.

Let us first consider the first problem. From (5) we can obtain the following expressions for the components of perturbing acceleration:

$$R^{(\mathrm{p})} = -K_2^{(\mathrm{p})} K_{\mathrm{p}} \frac{\bar{a}^5 a^3}{r^8} n^2 \left[\frac{2\mathbf{r}(\mathbf{r}\mathbf{v})}{r^2} + [\mathbf{r}\Omega] + \mathbf{v} \right]_R^{(\mathrm{p})}, \tag{10}$$

$$T^{(\mathrm{p})} = -K_2^{(\mathrm{p})} K_{\mathrm{p}} \frac{\bar{a}^5 a^3}{r^8} n^2 \left[\frac{2\mathbf{r}(\mathbf{r}\mathbf{v})}{r^2} + [\mathbf{r}\Omega] + \mathbf{v} \right]_T^{(\mathrm{p})}. \tag{11}$$

The upper index (p) indicates that the expression is used in the problem of tides raised on planet's body. Here and below, the lower indices R and T denote corresponding components of vectors.

It is obvious that the first term in square brackets has only radial component, the second one has only tangential component, and the third one has both. We assume that the satellite's orbital plane is normal to the vector of planet's angular rotation $\mathbf{\Omega}$. Hence the vector $[\mathbf{r\Omega}]$ lies in the orbital plane, it is normal to the vector $\mathbf{r}$ and points to the direction opposite to that of satellite's motion. To get the radial components, we used the fact that, for an arbitrary vector $\mathbf{V}$, its radial component can be obtained from the expression $(\mathbf{V}, \mathbf{r})/r$.

From the formulae of Keplerian motion we have

$$r = \frac{a(1-e^2)}{1+e\cos f}, \; \mathbf{v}_R = \frac{an}{\sqrt{1-e^2}} e \sin f,$$

$$\mathbf{v}_T = \frac{an}{\sqrt{1-e^2}}(1+e\cos f), \; (\mathbf{r}\mathbf{v}) = \frac{ane}{\sqrt{1-e^2}} r \sin f.$$

Taking into account these relationships, we obtain

$$\left[\frac{2\mathbf{r}(\mathbf{r}\mathbf{v})}{r^2} + [\mathbf{r}\Omega] + \mathbf{v} \right]_R^{(\mathrm{p})} = 3\frac{nae}{\sqrt{1-e^2}} \sin f, \tag{12}$$

$$\left[\frac{2\mathbf{r}(\mathbf{r}\mathbf{v})}{r^2} + [\mathbf{r}\Omega] + \mathbf{v} \right]_T^{(p)} = \frac{na}{\sqrt{1-e^2}}(1+e\cos f) - \frac{a(1-e^2)}{1+e\cos f}|\Omega|. \tag{13}$$

Now, substituting relationships (12) and (13) into (10) and (11) and then substituting the results into (8) and (9), we have

$$\frac{\mathrm{d}a}{\mathrm{d}t} = K_2^{(\mathrm{p})} K_\mathrm{p} \frac{\bar{a}^5}{a^5} na \frac{2\sqrt{1-e^2}}{(1-e^2)^8}(1+e\cos f)^8 \times \left[|\Omega| - \frac{n}{(1-e^2)^{3/2}}(1+2e\cos f+3e^2-2e^2\cos^2 f)\right], \tag{14}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = K_2^{(\mathrm{p})} K_\mathrm{p} \frac{\bar{a}^5}{a^5} n \frac{\sqrt{1-e^2}}{(1-e^2)^8}(1+e\cos f)^8 \times \left\{|\Omega| \frac{1-e^2}{1+e\cos f}(\cos f+\cos E) - \frac{n}{\sqrt{1-e^2}}[3e\sin^2 f + (\cos f+\cos E)(1+e\cos f)]\right\}. \tag{15}$$

The derived equations exactly correspond to the initial equations (5) and (6) in rectangular coordinates. These equations are to be solved together with the equations for the argument of pericentre ω and mean anomaly M. Such a solution would correspond exactly to that of the equations in rectangular coordinates, since the orbital elements and the vectors of position and velocity remain interrelated by the formulae of Keplerian motion.

As demonstrated above, when in solution of the equations in coordinates the transformation is made from coordinates and velocities to the Keplerian elements, the changes in semimajor axis and eccentricity look like monotone evolving functions with superimposed short-period oscillations. These oscillations are rather small so that, in analysing satellite's orbital evolution, they can be neglected.

We suppose that by averaging the right-hand sides of equations (14) and (15) over time the solution of these equations will provide us with evolutionary changes in the elements free from short-period perturbations. It is possible to check this assumption by comparing the solutions of strict equations (5) and (6) in coordinates with those of averaged equations in elements. Identity of averaged solution of the equations in coordinates with the solution of averaged equations in elements would allow us to study the long-term orbital evolution of satellites caused by tidal friction in planetary and satellite bodies by using only equations for a and e.

To make such a check, it is first necessary to derive averaged equations for the elements a and e and then to solve them by numerical integration. We derived such equations and obtained their solution.

When averaging equations (14) and (15), we had to make an expansion in powers of eccentricity neglecting the terms containing squared eccentricity. This simplification is acceptable because solution of the problem will supposedly be applied to the major moons of large planets whose orbital eccentricities are really small.

Now let us proceed to carrying out the procedures described above.

We denote averaged values by the bar above. In the process of averaging, the following relationships were used:

$$\overline{\cos f} = -e,\ \overline{\cos E} = -\frac{1}{2}e,\ \overline{\cos^2 f} = \frac{1}{2} + O(e^2),$$

$$\overline{\cos f\cos E} = \frac{1}{2} + O(e^2),$$

where $O(e^2)$ are terms of expansion in powers of e having the second order of smallness. In addition, we used expansions

$$(1+e\cos f)^k = 1 + ke\cos f + O(e^2),$$

where k is an arbitrary integer. Only the first two terms of the expansion were used.

At an intermediary stage of our actions we obtained the equations:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = 2K_2^{(\mathrm{p})} K_\mathrm{p} \frac{\bar{a}^5}{a^5} na(1+8e\cos f) \times \left[|\Omega| - n(1+2e\cos f+3e^2-2e^2\cos^2 f)\right], \tag{16}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = K_2^{(\mathrm{p})} K_\mathrm{p} \frac{\bar{a}^5}{a^5} n\,\{|\Omega|(1+7e\cos f)(\cos f+\cos E) - \frac{n}{\sqrt{1-e^2}}[3e\sin^2 f(1+8e\cos f) \tag{17}$$

$$+(\cos f+\cos E)(1+9e\cos f)]\}.$$

After averaging, we finally have

$$\frac{\mathrm{d}a}{\mathrm{d}t} = 2K_2^{(\mathrm{p})} K_\mathrm{p} \frac{\bar{a}^5}{a^5} na(|\Omega| - n),$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = K_2^{(\mathrm{p})} K_\mathrm{p} \frac{1}{2} \cdot \frac{\bar{a}^5}{a^5} \cdot (11|\Omega| - 18n)ne.$$

Then using first of the relations (7) we reduce the equations to the form

$$\frac{\mathrm{d}a}{\mathrm{d}t} = K_\mathrm{p} \frac{k_2}{Q_\mathrm{p}} \frac{\bar{a}^5}{a^5} na \tag{18}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = K_\mathrm{p} \frac{k_2}{Q_\mathrm{p}} \frac{1}{4} \cdot \frac{\bar{a}^5}{a^5} \cdot \frac{11|\Omega| - 18n}{|\Omega| - n} ne. \tag{19}$$

Our expressions (18) and (19) are in agreement with equations (9) and (10) from Hut (1981). They also agree with equations (41) and (46) from Kaula (1964), provided a misprint is corrected in the latter equation .[1]

Now let us look at how and which equations are obtained in the case of perturbing action of tides raised on viscoelastic body of satellite. From equations (6) we find that the radial and tangential components of acceleration have the form

$$R^{(\mathrm{s})} = -K_2^{(\mathrm{s})} K_\mathrm{s} \frac{\bar{a}^5 a^3}{r^8} n^2 \left[\frac{2\mathbf{r}(\mathbf{rv})}{r^2} + [\mathbf{r}\boldsymbol{\Omega}_\mathbf{s}] + \mathbf{v}\right]_R^{(\mathrm{s})}, \tag{20}$$

$$T^{(\mathrm{s})} = -K_2^{(\mathrm{s})} K_\mathrm{s} \frac{\bar{a}^5 a^3}{r^8} n^2 \left[\frac{2\mathbf{r}(\mathbf{rv})}{r^2} + [\mathbf{r}\boldsymbol{\Omega}_\mathbf{s}] + \mathbf{v}\right]_T^{(\mathrm{s})}. \tag{21}$$

Here, the upper index (s) indicates that the expression is used in the problem of tides raised on satellite's body. The lower indices R and T denote, as earlier, two components of vectors.

Supposing that the satellite's angular rotation rate $\boldsymbol{\Omega}_\mathbf{s}$ is normal to the orbital plane, similar to the previous case we have

$$\left[\frac{2\mathbf{r}(\mathbf{rv})}{r^2} + [\mathbf{r}\boldsymbol{\Omega}_\mathbf{s}] + \mathbf{v}\right]_R^{(\mathrm{s})} = 3\frac{nae}{\sqrt{1-e^2}}\sin f, \tag{22}$$

[1] A factor of 4, which is present in the first line of equation (46) from Kaula (1964), is missing in the second line of that equation.

$$\left[\frac{2\mathbf{r}(\mathbf{r}\mathbf{v})}{r^2} + [\mathbf{r}\Omega_\mathbf{s}] + \mathbf{v}\right]_T^{(s)}$$
$$= \frac{na}{\sqrt{1-e^2}}(1 + e\cos f) - \frac{a(1-e^2)}{1+e\cos f}|\Omega_\mathbf{s}|. \qquad (23)$$

Since we adopted the assumption that the satellite is in the state of constant synchronous rotation, we assume that $|\mathbf{\Omega_s}| = n$. Taking this into account and substituting (22) and (23) into (20) and (21) and then substituting the results into (8) and (9), we obtain

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -K_2^{(s)}K_s\frac{\bar{a}^5a^3}{r^8}\cdot\frac{2n^2a}{1-e^2}$$
$$\times\left[3e^2\sin^2 f + (1+e\cos f)^2 - (1-e^2)^{3/2}\right], \qquad (24)$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -K_2^{(s)}K_s\frac{\bar{a}^5a^3}{r^8}n^2\left\{3e\sin^2 f\right.$$
$$\left.+(\cos f + \cos E)\left[1 + e\cos f - \frac{(1-e^2)^{3/2}}{1+e\cos f}\right]\right\}. \qquad (25)$$

According to what was said earlier, we make simplifications in the right-hand sides of these equations, i.e. expand them in powers of eccentricity, leaving only main terms of expansion, and average them over time. At intermediary stage of this process, after expansion in powers of eccentricity, the following equations are obtained:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -K_2^{(s)}K_s\frac{\bar{a}^5}{a^5}2n^2a(1+8e\cos f)$$
$$\times\left(\frac{9}{2}e^2 - 2e^2\cos^2 f + 2e\cos f\right), \qquad (26)$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -K_2^{(s)}K_s\frac{\bar{a}^5}{a^5}n^2e[3\sin^2 f + 2\cos f(\cos f + \cos E)]. \qquad (27)$$

Averaging over time gives the final result:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -19K_2^{(s)}K_s\frac{\bar{a}^5}{a^5}n^2ae^2,$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{7}{2}K_2^{(s)}K_s\frac{\bar{a}^5}{a^5}n^2e.$$

Using the second of the relations (7) we transform these equations to the form

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -19K_s\frac{k_2^{(s)}}{Q_s}\frac{\bar{a}^5}{a^5}nae^2, \qquad (28)$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{7}{2}K_s\frac{k_2^{(s)}}{Q_s}\frac{\bar{a}^5}{a^5}ne. \qquad (29)$$

It is these equations that should describe the evolution of a and e caused by the dissipation of mechanical energy of satellite's orbital motion due to tidal friction in viscoelastic body of satellite itself.

Now we should compare the solution of the equations in rectangular coordinates that was obtained earlier with that of the equations (18) and (19) of the first problem and with that of the equations (28) and (29) of the second problem. We carried out numerical integration of the latter equations with the same initial conditions that were set in solving the differential equations in rectangular coordinates. These solutions are shown in the same Figs 3–6 and 7–10 corresponding to both problems. The lines of solutions coincide completely thus demonstrating exact identity (at least, within the limits of line's thickness) of both solutions. More accurate analysis proves that the solutions of the equations in orbital elements are exactly equal to the elements obtained from the solution of the equations in coordinates that were averaged to remove short-period oscillations.

This result proves our assumption that the solution of the equations (18), (19), (28), and (29) in the elements reliably describes satellite's orbital evolution in both problems.

Note that, for the second set of initial conditions, semimajor axis was chosen to satisfy the condition

$$n = \frac{11}{18}|\Omega|.$$

In this case, at the initial moment, the right-hand side of the equation (19) is equal to zero. The Fig. 4 shows that, in the beginning of the interval, the mean value of eccentricity almost does not change. It is this behaviour of this function that allows us to see short-period oscillations in the Fig. 2.

We note that the main conclusion of this paper remains valid even if $K_2^{(p)}$ is an arbitrary function of $\Omega - n$ and $K_2^{(s)}$ is an arbitrary function of n.

Differential equations in the Keplerian elements for the problem in consideration can be found in earlier papers. In particular, they are given in Lainey et al. (2012) where, for the case of tides on planet's body, reference is given to Kaula (1964) and, for the tides raised on satellite, authors refer to Peale & Cassen (1978). Using our notations, these equations have the form

$$\frac{\mathrm{d}a}{\mathrm{d}t} = K_\mathrm{p}\frac{k_2}{Q_\mathrm{p}}\frac{\bar{a}^5}{a^5}na, \qquad (30)$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = K_\mathrm{p}\frac{k_2}{Q_\mathrm{p}}\frac{57}{24}\cdot\frac{\bar{a}^5}{a^5}ne \ \text{(D)}, \qquad (31)$$

$$\frac{\mathrm{d}a}{dt} = -7K_s\frac{k_2^{(s)}}{Q_s}\frac{\bar{a}^5}{a^5}nae^2 \ \text{(D)}, \qquad (32)$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\frac{7}{2}K_s\frac{k_2^{(s)}}{Q_s}\frac{\bar{a}^5}{a^5}ne. \qquad (33)$$

Comparing these equations with the equations (18), (19), (28), and (29) obtained in our work makes it evident that two equations out of four coincide but two others (denoted by the letter D) significantly differ. To see the differences of the solutions graphically, we made plots (see the Figs 11 and 12). The differences in plots reflect significant differences in the solutions. The observed differences make it possible to conclude that the equations (31) and (32) given in Lainey et al. (2012) apparently follow from another model of tides that does not correspond to the formulae (1) and (2) taken from the same paper.

On the other hand, the equations in elements (31) and (32) do not correspond to the equations in coordinates (5) and (6). This means that if we had accepted that the equations (31) and (32) are correct, we would have to conclude that the equations (5) and (6) are wrong.

5 CONCLUSIONS

As a result of our work, we obtained the differential equations for evolution of semimajor axis and eccentricity of satellite's orbit caused by tidal friction in both planet's and satellite's bodies. The departing point for these equations were differential equations in rectangular coordinates given in Lainey et al. (2012). The averaged solutions of the equations in coordinates are proved to be identical with the precise solution of averaged equations in elements.

For the problem of tides raised on planet's body, the relationship was found between satellite's mean motion n and planet's rotation

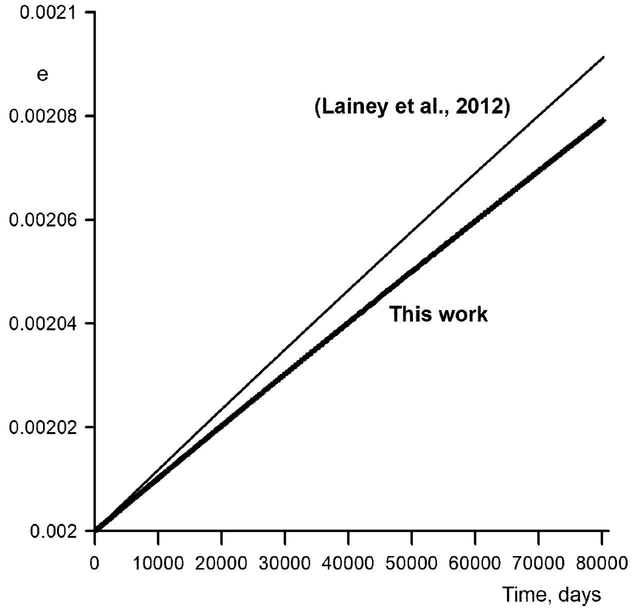

Figure 11. Change in eccentricity of satellite's orbit at 80200-d (220-yr) time interval caused by tidal friction in planet's body. The first set of initial conditions is used. The bold line corresponds to the solution of the equation obtained in this paper, the thin one corresponds to that of the equation given in Lainey et al. (2012).

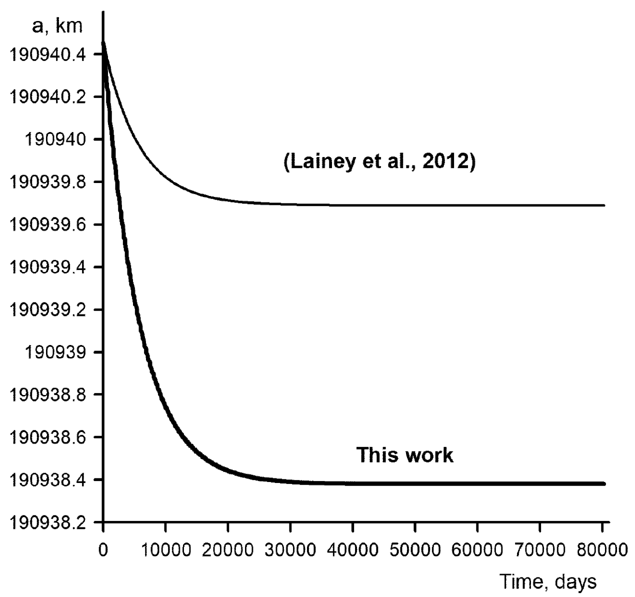

Figure 12. Change in semimajor axis of satellite's orbit at 80200-d (220-yr) time interval caused by tidal friction in satellite's body. The first set of initial conditions is used. The bold line corresponds to the solution of the equation obtained in this paper, the thin one corresponds to that of the equation given in Lainey et al. (2012).

rate $|\boldsymbol{\Omega}|$ when the rate of eccentricity's change becomes equal to zero. This happens when $n = \frac{11}{18}|\Omega|$.

The differential equations in the Keplerian elements obtained in this paper can be compared with those deduced by other authors. In total, there are four equations, two of them being those for semimajor axis and eccentricity in the problem of tides raised on planet's body. Two other equations are those for semimajor axis and eccentricity in the problem of tides raised on satellite. It turned out that the equations for semimajor axis for tides raised on planet and the equations for eccentricity for tides in satellite's body do coincide. However, two other equations do not coincide since the coefficients of equations are essentially different.

Note that it should be necessary to check the process of derivation of the equations in coordinates that we took from earlier works. The sources of these equations are the papers by Mignard (1979, 1980). We could not trace the derivation of these equations. It would also be necessary to check the correctness of generalization of the formulae in both papers by Mignard (1979, 1980) for the case of perturbations due to the tides raised on satellite's body that was made in Lainey et al. (2009b, 2012). It remains to be assumed that if initial equations in coordinates had been incorrect, all four equations in elements would have been different from those in other works. However, two out of four equations coincide, which gives us some confidence that initial equations in coordinates are correct.

The question remains open as to why two out of four equations in the Keplerian elements differ from those obtained by other authors. Since solutions of our four equations in the elements coincide with those of equations in coordinates, we infer that it is our equations in the elements that are correct. The equations obtained by other authors apparently describe some other process but not that of the change of satellite's Keplerian elements caused by the tides in viscoelastic bodies of planet and satellite and following from equations (5) and (6).

Note that our conclusions and results of numerical integration can be easily checked by anyone. The conclusions are given above, while numerical integration could be easily performed, all the equations being given in explicit form and initial numerical values for all necessary parameters being available.

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