Poisson brackets and truncations in nonlinear reduced fluid models for plasmas

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Abstract

The Hamiltonian structure for an infinite class of nonlinear reduced fluid models, derived from a Hamiltonian drift-kinetic system, is explicitly provided in terms of the $N+1$ fluid moments evolving in each model of the class, with N an arbitrary positive integer. This improves previous results, in which the existence of the Hamiltonian structure was shown, but the complete explicit expression for the Poisson bracket of each model of the class was not provided. We also show that, whereas the Hamiltonian functional of the fluid models can be derived from that of the drift-kinetic system, by projecting the perturbation of the distribution function onto its truncated series in terms of Hermite polynomials, this is not the case for the Poisson bracket. Indeed, the antisymmetric bilinear form obtained by means of the aforementioned projection, although, interestingly, "very similar" to the Poisson bracket of the fluid models, turns out to differ from it. The difference is found to reside in the coefficients $\mathbb{W}_{(N)\,l}^{mn}$ of the bilinear form, when the indices are such that $l+m+n$ is even and $l \geqslant N+1, m \geqslant N+1, n \geqslant N+1$. We show with a counterexample, related to the case $N=2$, that such bilinear form, in general, does not satisfy the Jacobi identity. We provide a physical interpretation of the set of variables $G_0, G_1, \cdots, G_N$, in terms of which the Poisson bracket of the fluid models exhibits a direct-sum structure, and point out an analogy between the present fluid reduction problem and the problem of the truncated quantum harmonic oscillator.

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1. Introduction

In many circumstances, the behaviour of plasmas is influenced by the presence of a magnetic field characterized by one dominant component, nearly constant in time, which is referred to as guide field component. This can be the case, for instance, in tokamaks or in coronal loops, where the toroidal component of the field is much greater than the poloidal component. In these situations, a small parameter naturally emerges, corresponding to the ratio between the characteristic amplitude of the time-dependent components of the magnetic field and the strong guide field component. Taking advantage of this small parameter, several nonlinear reduced fluid models have been derived in order to describe plasma dynamics in the presence of a strong guide field. In the context of tokamak fusion plasmas, two classical examples of such reduced models are provided by reduced magnetohydrodynamics (MHD) [1, 2] and by the Hasegawa-Wakatani model for drift-wave turbulence [3]. Reduced fluid models of the same kind also proved to be useful for investigating fundamental aspects of turbulence relevant for the solar wind, as for instance in Refs. [4, 5]. Further applications of reduced fluid models include the description of nonlinear coherent structures in plasmas [6, 7, 8] and magnetic reconnection [9, 10, 11, 12].

In the plasma physics literature, such nonlinear reduced fluid models were typically derived either from two-fluid models by asymptotic expansion in terms of small parameters (see, e.g. Refs. [1, 2, 13, 14, 15]), or by taking moments of gyrokinetic or drift-kinetic equations and imposing a closure relation (see, e.g. Refs. [16, 4, 17]).

From a dynamical systems perspective, a number of such models, when considered in their non-dissipative limit, were shown to possess a noncanonical Hamiltonian structure[18, 19, 15, 12, 20, 21, 22, 23, 24, 25, 26, 27, 28] (see also Ref. [29] for a review), which is the typical case for fluid models formulated from the Eulerian point of view [30]. The existence of a Hamiltonian structure is crucial for avoiding the presence of fake dissipative terms in the model, as well as for the opportunity it gives, to apply methods of Hamiltonian mechanics for the analysis of the dynamics described by the model [30, 31]. Qualitatively speaking, a Hamiltonian structure is identified by a phase space, a Poisson bracket acting on functions defined on the phase space, and a Hamiltonian, which is a prescribed function on the phase space. For the reduced fluid models under consideration, the phase space is a space of functions defined on the domain occupied by the plasma and satisfying appropriate boundary conditions. In the two-dimensional (2D) limit, where the dynamics is assumed to be invariant along the direction of the guide field, the noncanonical Poisson brackets for the reduced models are generally extensions [20] of the classical Lie-Poisson bracket of the 2D Euler equation for an incompressible fluid. The Poisson bracket for the full 3D models consists of the sum of the 2D bracket with a second Poisson bracket, such that some relations between the coefficients of the two brackets are satisfied, which guarantees the Jacobi identity [23]. The Hamiltonian of the models, on the other hand, typically consists of a functional on the phase space, quadratic in the model field variables.

In Ref. [32] it was shown how an infinite class of Hamiltonian reduced fluid models can be obtained, by imposing a particular closure on the hierarchy of fluid equations obtained by taking moments of a drift-kinetic system. In Ref. [33], this result was extended to more general models accounting also for finite Larmor radius effects, equilibrium temperature anisotropy and magnetic fluctuations along the direction of the guide field. However, such results still suffered from a gap, that we now briefly describe. Indeed, for a fluid model evolving $N+1$ moments of a given particle species, the expression of the Poisson bracket provided in Refs. [32, 33] in terms of the fluid moments (see in particular Eq. (81) of Ref. [32]), depends on a set of real numbers $\lambda_0, \lambda_1, \cdots, \lambda_N$, corresponding to the eigenvalues of an explicitly given, symmetric matrix denoted as W (according to the notation of Ref. [32]). The expression of the Poisson bracket also depends on an orthogonal matrix U, thanks to which, the matrix W can be put in diagonal form, according to the relation $U^T W U = \mathrm{diag}(\lambda_0, \lambda_1, \cdots, \lambda_N)$, where U^T is the transpose of U and $\mathrm{diag}(\lambda_0, \lambda_1, \cdots, \lambda_N)$ is the diagonal matrix with elements $\lambda_0, \lambda_1, \cdots, \lambda_N$ on the main diagonal. The knowledge of the matrix

U and of the eigenvalues of W is also necessary to cast the Poisson bracket in its simplest form (putting in evidence its direct sum structure) which occurs when the bracket is expressed in terms of a set of variables $G_0, G_1, \cdots, G_N$, alternative to the fluid moments. Although the existence of such an orthogonal matrix U is guaranteed by the spectral theorem, its explicit expression, for arbitrary N, was not provided in Refs. [32, 33]. Likewise, the properties of the eigenvalues of W of being real is guaranteed by the spectral theorem, but an explicit formula for their expression was not given. Therefore, although the Hamiltonian structure for the class of fluid models under consideration was shown to exist, the actual expression for the Poisson bracket, for a given model, had to be found by determining, case by case, the matrix U and the eigenvalues $\lambda_0, \lambda_1, \cdots, \lambda_N$. Of course, this deficiency limited the immediate applicability of the results of Refs. [32, 33], if not for the cases with very low N, where only the evolution of the first few fluid moments is retained. We recall [32] that the matrix U actually provides the transformation from the variables $G_0, G_1, \cdots, G_N$ to the fluid moments. Indeed, if we indicate the fluid moments with $g_0, g_1, \cdots, g_N$, such transformation is given by $g_m = U_{mn} G_n$ (see also Eq. (90) in the present paper, where the notation has been changed with respect to Ref. [32] and the matrix U is indicated with U_N).

One of the purposes of the present paper is to remedy the above mentioned gap, by providing the explicit expression for the Poisson bracket for any model belonging to the class treated in Refs. [32, 33]. This is made possible by making use of actually rather elementary properties of Jacobi matrices and Hermite polynomials.

A second objective of this paper concerns the relation between the Hamiltonian structure of the reduced fluid models and that of the parent drift-kinetic (or gyro-kinetic) model. Indeed, the parent model describes the evolution of the generalized perturbed distribution function $g(x, y, z, v, t)$, given by a linear combination of the perturbation of the gyrocenter distribution function with the component of the magnetic vector potential along the guide field (see Eq. (5)). The function g depends on the spatial coordinates x, y, z, as well as on time

t and on the component v of the velocity along the guide field. We consider generalized perturbed distribution functions that can be expanded in series as

$$g(x,y,z,v,t) = \sum_{n=0}^{+\infty} g_n(x,y,z,t)\frac{H_n(v)}{\sqrt{n!}}F_{eq}(v), \tag{1}$$

where F_{eq} is a Maxwellian distribution function, H_n is the Hermite polynomial of order n and the coefficients g_n correspond to the fluid moments. The evolution of g is then governed by the infinite system of evolution equations for the moments g_n, obtained by projecting the Hamiltonian drift-kinetic (or gyrokinetic) equation for g on the basis of Hermite polyomials. The $(N+1)$-moment Hamiltonian reduced fluid models, on the other hand, are obtained by imposing that the index n in the series (1) goes from 0 to $N < +\infty$, which truncates the infinite system of equations of the parent model, reducing it to a closed set of $N+1$ equations. The natural question then arises, about whether the Hamiltonian and the Poisson bracket of a reduced fluid model can be derived from the Hamiltonian and the Poisson bracket of the parent model, by replacing, in the latter, the perturbation of the distribution function, given by the infinite series (1), with the series truncated at the moment of order N. The analysis we describe in the present paper shows that, whereas this occurs for the Hamiltonian, it is not the case for the Poisson bracket. In particular, it is the 2D component of the Poisson bracket for the fluid model, which turns out to differ from the corresponding bilinear form, obtained from the Poisson bracket of the parent model upon replacing the perturbed distribution function with its truncated series. Roughly speaking, the Poisson bracket and the bilinear form turn out to be "very similar" and the difference concerns some coefficients which, in a sense that will be made precise later, are associated with "high-order" moments, among those retained in each fluid model. Moreover, by means of a counterexample, we show that, in general, the bilinear form obtained from the truncated series, does not satisfy the Jacobi identity, and thus is not a Poisson bracket.

With the present paper, we also provide a physical interpretation of the alternative variables $G_0, G_1, \cdots, G_N$. Indeed, making use of properties of Hermite polynomials, we show that such variables are proportional to what we refer to as the truncated generalized perturbed distribution function, evaluated at values of the parallel velocity coordinate equal to $\lambda_0, \lambda_1, \cdots, \lambda_N$. This might also help to shed light on previous considerations on the phenomenon of magnetic reconnection, based on the dynamics of such alternative variables in the case $N = 1$ [34, 35, 36].

A final objective of this paper is to point out an analogy between the infinite hierarchy of fluid equations obtained from the parent drift-kinetic system and the problem of a quantum harmonic oscillator. In particular, the closure problem shares similarities with the problem of the truncated quantum harmonic oscillator.

For the sake of simplicity, in the present paper the results are illustrated considering, as parent model, a relatively simple drift-kinetic model for the electron dynamics. However, the results can be extended to the more refined hybrid and gyrokinetic parent models considered in Refs. [32, 33].

Again for the sake of simplicity, we restrict to the case of a bounded spatial domain, where the fluid moments satisfy periodic boundary conditions. This allows for a simple derivation of the explicit expressions for the operators ϕ_{dk}, A_{dk}, ϕ_{fl}, A_{fl} (see Eqs. (29), (33), (36) and the immediately subsequent formulas), relating the generalized perturbed distribution function and the fluid moments, with the electromagnetic potentials. Besides the argument of simplicity, periodic boundary conditions are, in any event, of some relevance, as they are often adopted in numerical simulations of reduced fluid models. On the other hand, as will be discussed at the end of Sec. 3.1, the choice of periodic boundary conditions requires some restrictions on the set of observables on the phase space.

The paper is organized as follows. In Sec. 2 we introduce the parent drift-kinetic model. Section 3 describes the Hamiltonian structure of the parent drift-kinetic model and of the reduced fluid models obtained after imposing a Hamiltonian closure provided in Ref. [32]. The latter structure is expressed in terms of the variables $G_0, G_1, \cdots, G_N$. This Section essentially reviews already known results but formulates them in a more precise manner, with respect to Ref. [32]. In Sec. 4 we present one of our new results, consisting of the explicit general Hamiltonian structure of the reduced fluid models, in terms of the moments. In particular, with Proposition 4.1, the coefficients in the Poisson bracket are expressed in terms of Hermite polynomials and their zeros. Section 5 compares the Hamiltonian and the Poisson bracket of Sec. 4, with the functional and the bilinear form obtained from the Hamiltonian structure of the parent drift-kinetic model, by means of the aforementioned truncated series approach. The results of Secs. 4 and 5 are exemplified in Sec. 6, where the case $N = 2$ is treated in detail, also showing how the approach based on the truncated series can lead to a bilinear form which is not a Poisson bracket. In Sec. 7 a physical interpretation of the variables $G_0, G_1, \cdots, G_N$, as well as the analogy with the problem of the quantum harmonic oscillator are discussed. We conclude in Sec. 8. In AppendixA and AppendixB we provide the proofs of a Lemma and of a Proposition, respectively, formulated in Sec. 5.

2. Hamiltonian parent drift-kinetic model

We consider the following drift-kinetic model in normalized form

$$\frac{\partial g}{\partial t} + [\phi - vA, g] + v\frac{\partial}{\partial z}\left(g - \sqrt{\frac{2}{\beta_e}}F_{eq}(\phi - vA)\right) = 0, \tag{2}$$

$$\Delta_\perp \phi = \delta^2 \sqrt{\frac{2}{\beta_e}} \int_{-\infty}^{+\infty} dv\, g, \tag{3}$$

$$\Delta_\perp A - A = \sqrt{\frac{\beta_e}{2}} \int_{-\infty}^{+\infty} dv\, vg, \tag{4}$$

where Eq. (2) corresponds to the electron drift kinetic equation, whereas Eqs. (3) and (4) correspond to the quasi-neutrality relation and to the projection of Ampère's law along the direction of a magnetic guide field, respectively.

In Eqs. (2)-(4), the dynamical variable g is defined by

$$g(x, y, z, v, t) = f(x, y, z, v, t) - \sqrt{\frac{2}{\beta_e}} vF_{eq}(v)A(x, y, z, t), \tag{5}$$

where f is the actual perturbation of the electron gyrocenter distribution function, averaged with respect to the magnetic moment. We will refer to the function g as to the generalized perturbed distribution function. The field A is related to the normalized magnetic field $\mathbf{B}$ by

$$\mathbf{B}(x,y,z,t) = \nabla A(x,y,z,t) \times \hat{z} + \hat{z}, \tag{6}$$

where $\hat{z}$ is the unit vector along the z direction of a Cartesian coordinate system x, y, z. We note that the model assumes the presence of a strong uniform magnetic guide field along the z direction (corresponding to the second term on the right-hand side of Eq. (6)). The function F_{eq} is the Maxwellian equilibrium distribution function, whose explicit expression reads

$$F_{eq}(v) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{v^2}{2}}. \tag{7}$$

The field $\phi = \phi(x,y,z,t)$, on the other hand, corresponds to the electrostatic potential. The independent variables in Eqs. (2)-(4) are given by the spatial Cartesian coordinates x, y, and z, by the coordinate v, representing the velocity coordinate along the direction of the guide field, and by the time t. The spatial coordinates belong to the domain

$$\mathcal{D} = \{(x,y,z) \in \mathbb{R}^3 \mid -L_x \leqslant x \leqslant L_x, -L_y \leqslant y \leqslant L_y, -L_z \leqslant z \leqslant L_z\}, \tag{8}$$

with L_x, L_y and L_z positive real numbers. On the other hand, for the parallel velocity and time coordinates one has $-\infty < v < +\infty$ and $t \geqslant 0$, respectively. Periodic boundary conditions over the domain $\mathcal{D}$ are imposed on the fields g, A and ϕ, whereas we will assume that $g \to 0$ sufficiently fast, as $v \to \pm\infty$, in such a way that all integrals, with respect to v and involving g, converge.

Two parameters are present in the system and are defined as

$$\beta_e = 8\pi \frac{n_0 T_{0e}}{B_0^2}, \qquad \delta^2 = \frac{m_e}{m_i}, \tag{9}$$

where n_0 and T_{0e} are the uniform equilibrium particle density and electron temperature, respectively, with the temperature expressed in energy units. We indicated with B_0 the (dimensional) amplitude of the magnetic guide field, whereas m_e and m_i are the electron and ion mass, respectively. The perpendicular Laplacian operator $\Delta_\perp$ and the canonical Poisson bracket $[\ ,\ ]$, on the other hand, are defined by

$$\Delta_\perp f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}, \tag{10}$$

$$[f, g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}, \tag{11}$$

respectively, for two functions f and g.

As above anticipated, Eqs. (2)-(4) are expressed in terms of normalized quantities. The normalization of the adopted variables is given explicitly by

$$\begin{gathered} x = \frac{\tilde{x}}{d_e}, \qquad y = \frac{\tilde{y}}{d_e}, \qquad z = \frac{\tilde{z}}{L}, \qquad v = \frac{\tilde{v}}{v_{te}}, \qquad t = \frac{v_{te}}{L}\tilde{t}, \\ g = \frac{L}{d_e}\frac{v_{te}}{n_0}\tilde{g}, \qquad F_{eq} = \frac{v_{te}}{n_0}\tilde{F}_{eq}, \qquad \phi = \frac{L}{d_e}\frac{c\tilde{\phi}}{v_{te}B_0 d_e}, \qquad A = \frac{L}{d_e}\frac{\tilde{A}}{B_0 d_e}, \end{gathered} \tag{12}$$

where c is the speed of light, $d_e = c\sqrt{m_e/(4\pi e^2 n_0)}$ is the electron skin depth, L is a characteristic scale length of variation along the guide field direction, $v_{te} = \sqrt{T_{0e}/m_e}$ is the electron thermal speed. In the formulas (12), we denoted with a tilde the dimensional quantities.

The system (2)-(4) corresponds to a cold-ion, collisionless version of the electron drift-kinetic equation derived in Ref. [37]. The model belongs to the class of so-called δf models, which assume small fluctuations of the distribution functions, i.e. $\tilde{f}/\tilde{F}_{eq} \ll 1$. Weak spatial variations are assumed along the guide field direction, which corresponds to the condition $d_e/L \ll 1$. We also recall that the derivation of the model in Ref. [37] requires $\beta_e \ll 1$ and $\delta^2 \ll 1$. The model can be applied to describe, for instance, magnetic reconnection due to electron inertia in collisionless plasmas with a strong guide field, a phenomenon that can be relevant for tokamak devices and the solar corona. The same model was also adopted in Ref. [38] for the description of drift-Alfvén vortices in plasmas. In the same reference [38], the Hamiltonian structure of the model was also given.

We point out that, although the results presented in the present paper apply to the model (2)-(4), they can be extended to more sophisticated Hamiltonian parent models such as those treated in Refs. [32, 33], which account for further physical ingredients such as multiple species, parallel magnetic perturbations, finite Larmor radius effects and equilibrium temperature anisotropies. Our choice for the model (2)-(4) for the present article is mainly due to its relative simplicity, although, at the same time, the Hamiltonian structure of this system possesses all the fundamental features of the Hamiltonian structures of the more general parent models of Refs. [32, 33].

3. Hamiltonian structure of the parent drift-kinetic model and of the family of fluid models in terms of the variables $G_0, G_1, \cdots, G_N$

In this Section we review, although formulated in a more precise setting, the Hamiltonian structure of the parent model (2)-(4), as well as that of the family of fluid models that can be derived from it by means of a specific closure preserving the Hamiltonian structure. However, the Hamiltonian structure of the fluid models that we review in this Section, as anticipated in Sec. 1, is the one presented in Ref. [32], which is not expressed in terms of the fluid moments but in terms of alternative variables $G_0, G_1, \cdots, G_N$, which are linear combinations of the fluid moments, but with coefficients whose expressions are not known in general, so far.

Before proceeding with the review of such Hamiltonian structures, we find it appropriate to introduce some preliminary definitions. The purpose of the following Sec. 3.1 is to formulate a more precise setting, with respect to previous references such as Refs. [32, 33], for the drift-kinetic and the reduced fluid models. Also, we will recall a few notions, such as that of functional derivative, which will be repeatedly used throughout the paper. The readers already familiar with these subjects can of course skip these parts and go directly to Sec. 3.2.

3.1. Preliminaries

We first introduce the space $\mathcal{F}$ of smooth, periodic and square integrable functions on $\mathcal{D}$. This space will include the fluid Hermite moments of order greater than zero, and is given by

$$\mathcal{F} = \{h : \mathbb{R}^3 \to \mathbb{R} \,|\, h \in L^2(\mathcal{D}) \cap C^\infty(\mathcal{D}),$$
$$h(x,y,z) = h(x+2L_x,y,z) = h(x,y+2L_y,z) = h(x,y,z+2L_z) \quad \forall (x,y,z) \in \mathbb{R}^3\}. \tag{13}$$

Due to the periodicity assumption, an element $h \in \mathcal{F}$ can be represented as Fourier series in the following way:

$$h(\mathbf{x}) = \sum_{\mathbf{k}\in\mathscr{K}} h_{\mathbf{k}} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}, \tag{14}$$

where $\mathbf{x} = (x,y,z) \in \mathbb{R}^3$ and $\mathbf{k} = (k_x,k_y,k_z)$ is an element of the lattice $\mathscr{K}$ defined by

$$\mathscr{K} = \left\{ \left(\frac{\pi m}{L_x}, \frac{\pi n}{L_y}, \frac{\pi p}{L_z} \right),\, (m,n,p) \in \mathbb{Z}^3 \right\}. \tag{15}$$

In Eq. (14), the Fourier coefficients $h_{\mathbf{k}}$ are given by the Fourier transform

$$h_{\mathbf{k}} = \frac{1}{8L_xL_yL_z} \int_{\mathcal{D}} d^3x\, h(\mathbf{x}) \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{16}$$

It is now appropriate to introduce the subspace $\mathcal{F}_0$ of $\mathcal{F}$, consisting of the elements of $\mathcal{F}$ possessing zero spatial average with respect to x and y, over each plane z =constant. This will be the space of the fluid Hermite moment of order 0 and is given by

$$\mathcal{F}_0 = \{h \in \mathcal{F} \,|\, h_{\mathbf{k}} = 0 \quad \text{for}\, \mathbf{k} = (0,0,\pi p/L_z) \quad \text{with}\, p \in \mathbb{Z}\}. \tag{17}$$

The specificity of the moment of order 0 comes from the quasi-neutrality relation (3) which, as will be seen, implies that the moment of order 0 have zero spatial average in the plane perpendicular to the guide field.

Given a positive integer $N \in \mathbb{Z}_{>0}$, we also introduce the space

$$\mathcal{F}_N = \mathcal{F}_0 \times \underbrace{\mathcal{F} \times \cdots \times \mathcal{F}}_{N\ \text{times}}, \tag{18}$$

which will correspond to the phase space of a fluid model in which the highest-order moment retained in the evolution equations is the one of order N (note that such fluid model actually evolves in time $N+1$ moments, given that the lowest order moment is the one of order 0).

Next, we introduce the space $\mathcal{G}$ of the generalized perturbed distribution function:

$$\mathcal{G} = \Bigg\{ g : \mathbb{R}^4 \to \mathbb{R} \,|\, g(\mathbf{x}, v) = \sum_{n=0}^{+\infty} g_n(\mathbf{x}) \frac{H_n(v)}{\sqrt{n!}} F_{eq}(v),$$

$$g_0 \in \mathcal{F}_0, \quad g_i \in \mathcal{F} \quad \text{for}\, i \in \mathbb{Z}_{>0}, \quad H_n(v) = (-1)^n \mathrm{e}^{\frac{v^2}{2}} \frac{d^n}{dv^n} \mathrm{e}^{-\frac{v^2}{2}} \quad \text{for}\, n \in \mathbb{Z}_{\geqslant 0} \Bigg\}. \tag{19}$$

which will be the phase space of the parent drift-kinetic system. Note that, once the dynamical equations are introduced, and thus the time parameter t is added, the dependence on time of g is contained in the dependence on time of the coefficients g_n. According to a quite standard practice in δf drift-kinetic and gyrokinetic theory (see, for instance, Refs. [37, 39, 16]), we are considering generalized perturbed distribution functions that can be expressed as a series in Hermite polynomials H_n, multiplied by the equilibrium distribution function F_{eq}, which guarantees a sufficiently rapid decay of g as $v \to \pm\infty$. The orthogonality relation

$$\int_{-\infty}^{+\infty} dv\, H_m(v) H_n(v) F_{eq}(v) = n! \delta_{mn}, \tag{20}$$

permits to express the coefficients g_n of the expansion in Eq. (19), in terms of g, as

$$g_n = \frac{1}{\sqrt{n!}} \int_{-\infty}^{+\infty} dv\, H_n\, g, \qquad n \geqslant 0. \tag{21}$$

For given $g \in \mathcal{G}$ and $n \geqslant 0$, an element g_n defined by Eq. (21) will be referred to as *fluid moment*, or simply as *moment* of g of order n. We recall that the first Hermite polynomials correspond to $H_0 = 1, H_1 = v, H_2 = v^2 - 1, H_3 = v^3 - 3v, ...$ and that the first four moments are proportional to fluctuations of density, parallel canonical momentum, parallel temperature and parallel heat flux, respectively, of the electron gyrocenters. In particular, following Eq. (21), one has

$$g_0 = \int_{-\infty}^{+\infty} dv\, g, \qquad g_1 = \int_{-\infty}^{+\infty} dv\, vg, \tag{22}$$

$$g_2 = \frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} dv\, (v^2 - 1) g, \qquad g_3 = \frac{1}{\sqrt{6}} \int_{-\infty}^{+\infty} dv\, (v^3 - 3v) g, \cdots \tag{23}$$

Evidently, any element $g \in \mathcal{G}$ also admits a representation in Fourier series, with respect to $\mathbf{x}$, according to:

$$g(\mathbf{x}, v) = \sum_{\mathbf{k} \in \mathscr{K}} \tilde{g}_{\mathbf{k}}(v) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}, \tag{24}$$

with $\mathbf{x} \in \mathbb{R}^3$, $v \in \mathbb{R}$ and where the Fourier coefficients $\tilde{g}_{\mathbf{k}}$, following the expressions (16) (although the tilde symbol in this case was also added, in order to avoid possible confusion with the fluid moments g_n) and (19), read

$$\tilde{g}_{\mathbf{k}} = \sum_{n=0}^{+\infty} \frac{g_{n_{\mathbf{k}}}}{\sqrt{n!}} H_n F_{eq}, \qquad \mathbf{k} \in \mathscr{K}. \tag{25}$$

Equations

$$\Delta_\perp \phi = \delta^2 \sqrt{\frac{2}{\beta_e}} \int_{-\infty}^{+\infty} dv\, g, \tag{26}$$

$$\Delta_\perp A - A = \sqrt{\frac{\beta_e}{2}} \int_{-\infty}^{+\infty} dv\, vg, \tag{27}$$

can be solved in Fourier space with respect to ϕ and A, for a given drift-kinetic generalized perturbed distribution function $g \in \mathcal{G}$. The solutions for the electromagnetic potentials in terms of drift-kinetic generalized perturbed distribution function (for which we use the subscript dk) are two elements $\phi, A \in \mathcal{F}$ given by

$$\phi = \phi_{dk}[g], \qquad A = A_{dk}[g], \tag{28}$$

with $\phi_{dk}[\,] : \mathcal{G} \to \mathcal{F}$ and $A_{dk}[\,] : \mathcal{G} \to \mathcal{F}$ linear operators acting on $g \in \mathcal{G}$ in the following way:

$$\phi_{dk}[g](\mathbf{x}) = \sum_{\mathbf{k} \in \mathscr{K}} \phi_{dk}[g]_{\mathbf{k}} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}, \qquad A_{dk}[g](\mathbf{x}) = \sum_{\mathbf{k} \in \mathscr{K}} A_{dk}[g]_{\mathbf{k}} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}, \tag{29}$$

where

$$\phi_{dk}[g]_{\mathbf{k}} = -\delta^2 \sqrt{\frac{2}{\beta_e}} \int_{-\infty}^{+\infty} dv\, \frac{\tilde{g}_{\mathbf{k}}}{k_\perp^2}, \qquad \text{for} \quad \mathbf{k} \in \mathscr{K} \setminus \{(0, 0, \pi p/L_z)\,,\, p \in \mathbb{Z}\}\,, \tag{30}$$

$$\phi_{dk}[g]_{\mathbf{k}} = \phi_{0_p}, \qquad \text{for} \quad \mathbf{k} \in \{(0, 0, \pi p/L_z)\,,\, p \in \mathbb{Z}\}\,, \tag{31}$$

$$A_{dk}[g]_{\mathbf{k}} = -\sqrt{\frac{\beta_e}{2}} \int_{-\infty}^{+\infty} dv\, v \frac{\tilde{g}_{\mathbf{k}}}{1 + k_\perp^2}, \qquad \text{for} \quad \mathbf{k} \in \mathscr{K}. \tag{32}$$

In Eq. (31), $\phi_{0_p} \in \mathbb{C}$, for $p \in \mathbb{Z}$, is a family of complex constants, such that $\phi_{0_n} = \bar{\phi}_{0_{-n}}$ for $n \in \mathbb{Z}_{>0}$, with the overbar indicating the complex conjugate. The choice of the arbitrary constants ϕ_{0_n}, for $n \in \mathbb{Z}_{\geqslant 0}$, fixes the z-dependence of the value of the spatial average of ϕ, with respect to x and y, on planes z =constant.

In Eqs. (30)-(32) we also introduced the perpendicular wave number $k_\perp$ defined as $k_\perp = \sqrt{k_x^2 + k_y^2}$.

The solutions (30)-(32) permit then to express, at any time t, the electromagnetic potentials ϕ and A in terms of g in Eqs. (3)-(4).

Using Eq. (25) and the orthogonality relation (20), one can also express the solutions for ϕ and A as the images of linear operators (with the subscript fl to indicate that ϕ and A, in this case, are expressed in terms of fluid variables) $\phi_{fl}[\,] : \mathcal{F}_0 \to \mathcal{F}$ and $A_{fl}[\,] : \mathcal{F} \to \mathcal{F}$, acting on the fluid moments g_0 and g_1, respectively. More precisely, if we define, for $u \in \mathcal{F}_0$ and $w \in \mathcal{F}$:

$$\phi_{fl}[u](\mathbf{x}) = \sum_{\mathbf{k} \in \mathscr{K}} \phi_{fl}[u]_{\mathbf{k}} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}, \tag{33}$$

where

$$\phi_{fl}[u]_{\mathbf{k}} = -\delta^2 \sqrt{\frac{2}{\beta_e}} \frac{u_{\mathbf{k}}}{k_\perp^2}, \qquad \text{for} \quad \mathbf{k} \in \mathscr{K} \setminus \{(0, 0, \pi p/L_z)\,,\, p \in \mathbb{Z}\}\,, \tag{34}$$

$$\phi_{fl}[u]_{\mathbf{k}} = \phi_{0_p}, \qquad \text{for} \quad \mathbf{k} \in \{(0, 0, \pi p/L_z)\,,\, p \in \mathbb{Z}\}\,, \tag{35}$$

and

$$A_{fl}[w](\mathbf{x}) = \sum_{\mathbf{k} \in \mathscr{K}} A_{fl}[w]_{\mathbf{k}} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}, \tag{36}$$

with

$$A_{fl}[w]_{\mathbf{k}} = -\sqrt{\frac{\beta_e}{2}} \frac{w_{\mathbf{k}}}{1 + k_\perp^2}, \qquad \text{for} \quad \mathbf{k} \in \mathscr{K}\,, \tag{37}$$

one has, for $g \in \mathcal{G}$, the relations

$$\phi_{dk}[g]_{\mathbf{k}} = -\delta^2 \sqrt{\frac{2}{\beta_e}} \sum_{n=0}^{+\infty} \frac{1}{\sqrt{n!}} \frac{g_{n_{\mathbf{k}}}}{k_\perp^2} \int_{-\infty}^{+\infty} dv\, H_n F_{eq}$$

$$= -\delta^2 \sqrt{\frac{2}{\beta_e}} \frac{g_{0_{\mathbf{k}}}}{k_\perp^2} = \phi_{fl}[g_0]_{\mathbf{k}}, \qquad \text{for} \quad \mathbf{k} \in \mathscr{K} \setminus \{(0,0,\pi p/L_z)\,,\, p \in \mathbb{Z}\}\,, \tag{38}$$

$$\phi_{dk}[g]_{\mathbf{k}} = \phi_{0_p} = \phi_{fl}[g_0]_{\mathbf{k}}, \qquad \text{for} \quad \mathbf{k} \in \{(0,0,\pi p/L_z)\,,\, p \in \mathbb{Z}\}\,, \tag{39}$$

$$A_{dk}[g]_{\mathbf{k}} = -\sqrt{\frac{\beta_e}{2}} \sum_{n=0}^{+\infty} \frac{1}{\sqrt{n!}} \frac{g_{n_{\mathbf{k}}}}{1+k_\perp^2} \int_{-\infty}^{+\infty} dv\, v H_n F_{eq}$$

$$= -\sqrt{\frac{\beta_e}{2}} \frac{g_{1_{\mathbf{k}}}}{1+k_\perp^2} = A_{fl}[g_1]_{\mathbf{k}}, \qquad \text{for} \quad \mathbf{k} \in \mathscr{K}. \tag{40}$$

where, in Eqs. (38)-(40), we made use of Eqs. (22), (25) and of the orthogonality relation (20). The electrostatic and magnetic potentials ϕ and A can thus be expressed in terms of the zeroth and first order moments of g by

$$\phi = \phi_{fl}[g_0], \qquad A = A_{fl}[g_1], \tag{41}$$

respectively.

The condition for g_0 of having zero spatial average on planes z =constant comes from the fact that Eq. (3), for each $\mathbf{k} \in \mathscr{K}$, implies $-(k_x^2 + k_y^2)\phi_{\mathbf{k}} = \delta^2 \sqrt{2/\beta_e} g_{0_{\mathbf{k}}}$. When evaluated at $\mathbf{k} = (0,0,\pi p/L_z)$, for $p \in \mathbb{Z}$ and $\phi_{(0,0,\pi p/L_z)} \neq 0$, this relation implies $g_{0_{(0,0,\pi p/L_z)}} = 0$. The electrostatic potential ϕ, on the other hand, is determined up to the choice of the arbitrary constants ϕ_{0_n}, with $n \in \mathbb{N}$.

In order to introduce the Hamiltonian structures of the drift-kinetic and fluid models, it is also convenient to define here the functional derivatives that we will make use of, later in the paper.

Given a real functional $F : \mathcal{G} \to \mathbb{R}$, we denote its functional derivative, with respect to $g \in \mathcal{G}$, as $\delta F/\delta g$ and we define it by means of the relation

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}(F(g + \epsilon \delta g) - F(g)) = \int_{-\infty}^{+\infty} dv \int_{\mathcal{D}} d^3x\, \delta g \frac{\delta F}{\delta g}, \tag{42}$$

for all $\delta g \in \mathcal{G}$. Functional derivatives of this type will appear in the Hamiltonian formulation of the drift-kinetic model.

With regard to the functional derivatives with respect to the moments, occurring in fluid models, given a functional $F : \mathcal{F}_N \to \mathbb{R}$, its functional derivative with respect to $h \in \mathcal{F}_N$ is denoted as $\delta F/\delta h$ and corresponds to $\delta F/\delta h = (\delta F/\delta h_0, \delta F/\delta h_1, \cdots, \delta F/\delta h_N)$ defined by

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}(F(h + \epsilon \delta h) - F(h)) = \int_{\mathcal{D}} d^3x \, \delta h \cdot \frac{\delta F}{\delta h}, \tag{43}$$

for all $\delta h = (\delta h_0, \delta h_1, \cdots, \delta h_N) \in \mathcal{F}_N$. Note that the dot on the right-hand side of Eq. (43) denotes a scalar product, so that $\delta h \cdot \delta F/\delta h = \sum_{i=0}^{N} \delta h_i (\delta F/\delta h_i)$. Moreover, by varying one δh_i at the time, for $i = 0, 1, \cdots, N$, while keeping $\delta h_j = 0$ for $j \neq i$, one obtains

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}(F(h_0, \cdots, h_i + \epsilon \delta h_i, \cdots, h_N) - F(h_0, \cdots, h_i, \cdots, h_N)) = \int_{\mathcal{D}} d^3x \, \delta h_i \frac{\delta F}{\delta h_i}, \tag{44}$$

which singles out the expression of $\delta F/\delta h_i$, defining the functional derivative of F with respect to h_i, for a given i.

As will be reviewed in Sec. 3.2, the Poisson bracket of a Hamiltonian system acts on observables, which are functions defined on the phase space. In the case of the parent drift-kinetic model and of the reduced fluid models derived from it, the Jacobi identity for the Poisson bracket relies on the relation

$$\int_{\mathcal{D}} d^3x \, u[v, w] = \int_{\mathcal{D}} d^3x \, w[u, v], \tag{45}$$

for functions $u, v, w \in \mathcal{F}$. The periodic boundary conditions satisfied by u, v and w imply that boundary terms vanish when integrating by parts, which is required to prove the identity (45). Because the Poisson bracket for the drift-kinetic model and the reduced fluid models will contain functional derivatives of observables, at the place of v and w, in an expression analogous to the left-hand side of Eq. (45) (see, for instance Eqs. (50) and (87)), we require the observables to be such that their functional derivatives satisfy periodic boundary conditions on $\mathcal{D}$.

We will indicate with $\bar{C}^{\infty}(\mathcal{P})$ the set of real smooth functionals over a phase space $\mathcal{P}$, such that the functional derivatives of these functionals satisfy periodic boundary conditions on $\mathcal{D}$. So, for instance, the set of observables for the parent drift-kinetic model will be denoted as $\bar{C}^{\infty}(\mathcal{G})$, whereas the set of observables for the $(N+1)$-moment fluid models will be $\bar{C}^{\infty}(\mathcal{F}_N)$.

3.2. Hamiltonian formulation of the parent drift-kinetic model

Based on Refs. [38, 32], we review here the (noncanonical) Hamiltonian structure of the parent drift-kinetic model (2)-(4). First, we recall that (see, e.g. Refs. [30, 31, 40]) a Hamiltonian system on a phase space $\mathcal{P}$ is a dynamical system

$$\dot{u}(t) = J(u(t))\partial_u H(u(t)), \tag{46}$$

where the dot indicates the time derivative and $u : \mathbb{R} \to \mathcal{P}$ is a curve describing the time evolution in phase space of the dynamical variable u following an initial condition $u(0) = u_0 \in \mathcal{P}$. We denote with $\Phi(\mathcal{P})$ the set of observables of the dynamical system. On the right-hand side of Eq. (46) the symbol J indicates the Poisson operator $J : T^*\mathcal{P} \to T\mathcal{P}$ (where $T^*\mathcal{P}$ and $T\mathcal{P}$ are the cotangent and tangent bundle of $\mathcal{P}$, respectively) and $H \in \Phi(\mathcal{P})$ is the observable corresponding to the Hamiltonian of the system. In Eq. (46) we also denoted with ∂_u a derivative with respect to u, which, in the infinite-dimensional case, takes the form of a functional derivative, as those defined in Eqs. (42) and (43).

A generic observable $F \in \Phi(\mathcal{P})$ of the Hamiltonian system (46) evolves according to

$$\dot{F} = \{F, H\}, \tag{47}$$

where $\{\,,\}$ is a Poisson bracket, i.e. an antisymmetric bilinear form satisfying the Leibniz rule and the Jacobi identity. Note that, for two observables $F, G \in \Phi(\mathcal{P})$, the Poisson operator and the Poisson bracket are related by

$$\{F, G\} =< \partial_u F, J(u)\partial_u G >, \tag{48}$$

where $<\,,>$ indicates a dual pairing.

The Hamiltonian structure of a Hamiltonian system on a given phase space is thus determined by its Hamiltonian and its Poisson bracket. Due to the antisymmetry property of the bracket, it follows from (47) that $\dot{H} = 0$, which expresses the conservation of the total energy.

As anticipated in Sec. 3.1, the phase space for the parent drift-kinetic system is given by the set $\mathcal{G}$. The Hamiltonian structure of this system [38] consists of the following Hamiltonian functional $H_{dk} \in \bar{C}^{\infty}(\mathcal{G})$:

$$H_{dk}(g) = \frac{1}{2}\int_{\mathcal{D}} d^3x \int_{-\infty}^{+\infty} dv \, \left(\frac{g^2}{F_{eq}} - \sqrt{\frac{2}{\beta_e}} g(\phi - vA) \right), \tag{49}$$

where $\phi = \phi_{dk}[g]$, $A = A_{dk}[g]$, and of the Poisson bracket

$$\{F, G\}_{dk} = \int_{\mathcal{D}} d^3x \int_{-\infty}^{+\infty} dv \, \left(\sqrt{\frac{\beta_e}{2}} g[F_g, G_g] - vF_{eq}F_g \frac{\partial G_g}{\partial z} \right), \tag{50}$$

for $F, G \in \bar{C}^{\infty}(\mathcal{G})$. In Eq. (50) we also introduced the subscript notation on the functionals to indicate functional derivatives, so that, for instance $F_g = \delta F/\delta g$. Note that the antisymmetry of the second term in the Poisson bracket (50) relies on the fact that boundary terms vanish when integrating by parts.

We also observe that the Hamiltonian (49) physically corresponds to the sum of the energy associated with the small-amplitude fluctuations of distribution function around a Maxwellian equilibrium (first term on the right-hand side), with the energy associated with electromagnetic fluctuations, given by the remaining two terms on the right-hand side.

We proceed at a formal level, assuming that, for a generic initial condition $g(x, y, z, v, 0) = \mathrm{g}_0(x, y, z, v) \in \mathcal{G}$, the solution of the system (2)-(4) exists for $0 \leqslant t \leqslant T$, with $T > 0$, and that, for each solution, $g(x, y, z, v, t) \in \mathcal{G}$ for $0 \leqslant t \leqslant T$. We then identify a solution $g(x, y, z, v, t)$, corresponding to a given initial condition, with a curve $g : [0, T] \to \mathcal{G}$ that associates, at each time t, a point $g(t) \in \mathcal{G}$ in phase space.

Concerning the Hamiltonian structure of the drift-kinetic model, we remark that

$$\frac{\delta H_{dk}}{\delta g} = \frac{g}{F_{eq}} - \sqrt{\frac{2}{\beta_e}}(\phi_{dk}[g] - vA_{dk}[g]). \tag{51}$$

In order to derive the latter relation one makes use of the following symmetry properties of the operators $\phi_{dk}[\,]$ and $A_{dk}[\,]$:

$$\int_{\mathcal{D}} d^3x \int_{-\infty}^{+\infty} dv\, h\phi_{dk}[g] = \int_{\mathcal{D}} d^3x \int_{-\infty}^{+\infty} dv\, g\phi_{dk}[h], \tag{52}$$

$$\int_{\mathcal{D}} d^3x \int_{-\infty}^{+\infty} dv\, hA_{dk}[g] = \int_{\mathcal{D}} d^3x \int_{-\infty}^{+\infty} dv\, gA_{dk}[h], \tag{53}$$

for $g, h \in \mathcal{G}$. The properties (52)-(53) easily follow from the definitions (30)-(32) and from the orthogonality of the Fourier and Hermite bases.

From the expressions (48) and (50), using integration by parts, it follows that the Poisson operator associated with the drift-kinetic Poisson bracket (50) is given by

$$J_{dk}(g) = -\sqrt{\frac{\beta_e}{2}}[g, \cdot\,] - vF_{eq}\frac{\partial}{\partial z}, \tag{54}$$

with respect to the dual pairing

$$< f, h >= \int_{\mathcal{D}} d^3x \int_{-\infty}^{+\infty} dv\, fh, \tag{55}$$

for two functions f, h. Combining Eqs. (51) and (54) with the general expression (46) for a Hamiltonian system, one retrieves namely the drift-kinetic equation (2).

3.3. Hamiltonian formulation of the fluid models

Multiplying both sides of Eq. (2) by $H_n/\sqrt{n!}$, for $n = 0, 1, 2, \cdots$ and integrating over v, one obtains the following infinite system of fluid equations

$$\frac{\partial g_0}{\partial t} + [\phi, g_0] - [A, g_1] + \frac{\partial}{\partial z}\left(g_1 + \sqrt{\frac{2}{\beta_e}}A\right) = 0, \tag{56}$$

$$\frac{\partial g_1}{\partial t} + [\phi, g_1] - \sqrt{2}[A, g_2] - [A, g_0] + \frac{\partial}{\partial z}\left(\sqrt{2}g_2 + g_0 - \sqrt{\frac{2}{\beta_e}}\phi\right) = 0, \tag{57}$$

$$\frac{\partial g_2}{\partial t} + [\phi, g_2] - \sqrt{3}[A, g_3] - \sqrt{2}[A, g_1] + \frac{\partial}{\partial z}\left(\sqrt{3}g_3 + \sqrt{2}g_1 + \sqrt{2}\sqrt{\frac{2}{\beta_e}}A\right) = 0, \tag{58}$$

$$\vdots$$

$$\frac{\partial g_N}{\partial t} + [\phi, g_N] - \sqrt{N+1}[A, g_{N+1}] - \sqrt{N}[A, g_{N-1}] + \frac{\partial}{\partial z}\left(\sqrt{N+1}g_{N+1} + \sqrt{N}g_{N-1}\right) = 0, \tag{59}$$

$$\vdots$$

where

$$\Delta_\perp \phi = \delta^2 \sqrt{\frac{2}{\beta_e}} g_0, \tag{60}$$

$$\Delta_\perp A - A = \sqrt{\frac{\beta_e}{2}} g_1, \tag{61}$$

and the fluid moments $g_0, g_1, g_2, \cdots$ are defined in Eq. (21). We remark that, on the basis of our definitions, g_0, g_1 and g_2 are proportional to the fluctuations of the electron gyrocenter density, parallel velocity and parallel temperature, respectively, where 'parallel' refers to the direction of the guide field. Note also that the first three equations (56)-(58) of the hierarchy are peculiar, as they involve also derivatives, with respect to z, of the electromagnetic potentials ϕ and A. For $N > 2$, on the other hand, the equations of the hierarchy are given by Eq. (59).

Given a fixed integer $N \geqslant 1$, the infinite hierarchy of equations (56)-(59) can be truncated by imposing

$$g_{N+1} = 0. \tag{62}$$

The resulting closed system, given that we are assuming $g \in \mathcal{G}$ in the parent drift-kinetic system, can be written as

$$\begin{aligned} &\frac{\partial g_m}{\partial t} + [\phi, g_m] - S_{N_{mn}}[A, g_n] + S_{N_{mn}} \frac{\partial g_n}{\partial z} \\ &+ \sqrt{\frac{2}{\beta_e}} \frac{\partial}{\partial z} \left(\sqrt{m!} (\delta_{m0} + \delta_{m2}) A - \delta_{m1} \phi \right) = 0, \qquad m = 0, 1, \cdots, N, \end{aligned} \tag{63}$$

$$\Delta_\perp \phi = \delta^2 \sqrt{\frac{2}{\beta_e}} g_0, \tag{64}$$

$$\Delta_\perp A - A = \sqrt{\frac{\beta_e}{2}} g_1, \tag{65}$$

where the sum over repeated indices is understood and where $S_{N_{mn}}$ indicates the element of row m and column n, of the $(N+1)\times(N+1)$ matrix

$$S_N = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & \sqrt{2} & 0 & \dots & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \dots & 0 \\ 0 & 0 & \sqrt{3} & 0 & \dots & 0 \\ & \dots & & & \dots & \\ & \dots & & & \dots & \\ 0 & 0 & 0 & \dots & 0 & \sqrt{N} \\ 0 & 0 & 0 & \dots & \sqrt{N} & 0 \end{pmatrix}. \tag{66}$$

We specify that, in this paper, the indices of the rows and of the columns of $(N+1)\times(N+1)$ matrices run from 0 to N.

Because the matrix S_N is real symmetric, it exists an orthogonal matrix $U_N \in O(N+1)$ such that $U_N^T S_N U_N = \mathrm{diag}(\lambda_0, \lambda_1, \cdots, \lambda_N)$, where $\lambda_0, \lambda_1, \cdots, \lambda_N$ are the eigenvalues of S_N. One can then introduce the alternative variables

$$G_i = U^T_{N_{im}} g_m, \qquad i = 0, 1, \cdots, N, \tag{67}$$

in terms of which the system (63)-(65) can be rewritten as

$$\frac{\partial G_i}{\partial t} + [\phi - \lambda_i A, G_i] + \lambda_i \frac{\partial G_i}{\partial z} + \sqrt{\frac{2}{\beta_e}} \frac{\partial}{\partial z} \left((U^T_{N_{i0}} + \sqrt{2!} U^T_{N_{i2}}) A - U^T_{N_{i1}} \phi \right) = 0, \qquad i = 0, 1, \cdots, N, \tag{68}$$

$$\Delta_\perp \phi = \delta^2 \sqrt{\frac{2}{\beta_e}} \sum_{m=0}^{N} U_{N_{0m}} G_m, \tag{69}$$

$$\Delta_\perp A - A = \sqrt{\frac{\beta_e}{2}} \sum_{m=0}^{N} U_{N_{1m}} G_m. \tag{70}$$

In Ref. [32] it was shown that the system (68)-(70) (which is equivalent to Eqs. (63)-(65)) is Hamiltonian, with Hamiltonian functional

$$\mathcal{H}(G_0, G_1, \cdots, G_N) = \frac{1}{2} \int_{\mathcal{D}} d^3x \left(\sum_{n=0}^{N} G_n^2 - \sqrt{\frac{2}{\beta_e}} U_{N_{0i}} G_i \left(\phi_{fl}[U_{N_{0l}} G_l] - \lambda_i A_{fl}[U_{N_{1l}} G_l] \right) \right), \tag{71}$$

where, due to the orthogonality of the matrix U_N, from Eq. (67), one has $U_{N_{0l}} G_l = g_0$ and $U_{N_{1l}} G_l = g_1$ (we recall that, also in the expression (71), the sum over repeated indices is understood). In Eq. (71) we are considering $\mathcal{H} \in \bar{C}^{\infty}(\mathcal{G}_N)$, where

$$\mathcal{G}_N = \{(G_0, G_1, \cdots, G_N) \in \mathcal{F} \times \mathcal{F} \times \cdots \times \mathcal{F} \mid \sum_{l=0}^{N} U_{N_{0l}} G_l \in \mathcal{F}_0\}. \tag{72}$$

The introduction of this set is required in order for the expression $\phi_{fl}[U_{N_{0l}} G_l]$ to be well defined and, in turn, for the equation (69) to be solved with respect to ϕ.

The Poisson bracket, on the other hand, is given by

$$\{F, K\}_G = \{F, K\}_{G_\perp} + \{F, K\}_{G_\parallel}, \tag{73}$$

where

$$\{F, K\}_{G_\perp} = \sqrt{\frac{\beta_e}{2}} \sum_{i=0}^{N} \frac{1}{U_{N_{0i}}} \int_{\mathcal{D}} d^3x\, G_i [F_{G_i}, K_{G_i}],$$

$$\{F, K\}_{G_\parallel} = -\sum_{i=0}^{N} \lambda_i \int_{\mathcal{D}} d^3x F_{G_i} \frac{\partial K_{G_i}}{\partial z},$$

for $F, K \in \bar{C}^{\infty}(\mathcal{G}_N)$. Using the relation

$$\frac{\delta \mathcal{H}}{\delta G_i} = G_i - \sqrt{\frac{2}{\beta_e}} U_{N_{0i}} (\phi_{fl}[U_{N_{0l}} G_l] - \lambda_i A_{fl}[U_{N_{1l}} G_l]), \qquad i = 0, 1, \cdots, N, \tag{74}$$

one can indeed obtain Eqs. (68)-(70), from Eqs. (71) and (73), applying the expression (46).

We remark that, in order to obtain the relation (74), we made use of the symmetry properties

$$\int_{\mathcal{D}} d^3x\, \eta \phi_{fl}[\xi] = \int_{\mathcal{D}} d^3x\, \xi \phi_{fl}[\eta], \tag{75}$$

$$\int_{\mathcal{D}} d^3x\, u A_{fl}[w] = \int_{\mathcal{D}} d^3x\, w A_{fl}[u], \tag{76}$$

for $\eta, \xi \in \mathcal{F}_0$ and $u, w \in \mathcal{F}$, which are a straightforward consequence of the relations (52) and (53).

4. Hamiltonian structure of the family of fluid models in terms of the moments $g_0, g_1, \cdots, g_N$

The Hamiltonian formulation of the fluid models described in Sec. 3.3 crucially depends on the knowledge of the matrix U_N and of the eigenvalues

$\lambda_0, \lambda_1, \cdots, \lambda_N$. The existence of such matrix is guaranteed by the spectral theorem and this is sufficient to show the existence of the Hamiltonian structure. However, in the absence of an explicit expression for the matrix U_N and for the eigenvalues of S_N, the general expression for the Hamiltonian (71) and the Poisson bracket (73) cannot be determined, and one is forced to find such matrix and such eigenvalues case by case, when possible, for a given N of interest. Moreover, the general expression of the transformation (67) is not known either. This expression is required in order to rewrite the Poisson bracket (73), for an arbitrary N, in terms of the fluid moments $g_0, g_1, \cdots, g_N$, which are the variables that one naturally adopts in applications.

In this Section we remedy this deficiency and provide the explicit expression for the matrix U_N and for the eigenvalues of S_N, as well as the Hamiltonian formulation of the fluid models (63)-(65), for arbitrary N, in terms of the moments $g_0, g_1, \cdots, g_N$.

Indeed, in Refs. [32, 33] it was not realized that the eigenvalues of the matrix S_N correspond to the zeros of the Hermite polynomial $H_{N+1}(x)$ (which is actually a well known fact, see, e.g. Ref. [41]). From the recurrence relation $xH_n(x) = H_{n+1}(x) + nH_{n-1}(x)$ it follows that

$$x\hat{H}_n(x) = \sqrt{n+1}\hat{H}_{n+1}(x) + \sqrt{n}\hat{H}_{n-1}(x), \tag{77}$$

where we defined

$$\hat{H}_n(x) = \frac{H_n(x)}{\sqrt{n!}}, \qquad n \geqslant 0. \tag{78}$$

Equation (77), evaluated at $x = \lambda_i$, for a given eigenvalue $\lambda_i \in \{\lambda_0, \lambda_1, \cdots, \lambda_N\}$, yields

$$\lambda_i\hat{H}_n(\lambda_i) = \sqrt{n+1}\hat{H}_{n+1}(\lambda_i) + \sqrt{n}\hat{H}_{n-1}(\lambda_i) \tag{79}$$

Combining Eq. (79) with the expression of S_N, one sees that the vector $(\hat{H}_0(\lambda_i), \hat{H}_1(\lambda_i), \cdots, \hat{H}_N(\lambda_i))^T$ is an eigenvector of S_N associated with the eigenvalue λ_i. Because the columns of the matrix U_N correspond to orthonormal eigenvectors of S_N, we have that a generic element of U_N is given by

$$U_{N_{mn}} = \frac{\hat{H}_m(\lambda_n)}{u_{(n)}}, \qquad m, n = 0, 1, \cdots, N, \tag{80}$$

where

$$u_{(n)} = \sqrt{\sum_{i=0}^{N} \hat{H}_i^2(\lambda_n)}, \qquad n = 0, 1, \cdots, N, \tag{81}$$

are normalization constants. The expression for these constants can be simplified making use of the Christoffel-Darboux identity [42]

$$\sum_{n=0}^{N} \frac{\bar{H}_n(x)\bar{H}_n(y)}{2^n n!} = \frac{\bar{H}_{N+1}(x)\bar{H}_N(y) - \bar{H}_N(x)\bar{H}_{N+1}(y)}{2^{N+1}N!(x-y)}, \tag{82}$$

where $\bar{H}_n(x) = 2^{\frac{n}{2}} H_n(\sqrt{2}x)$ are rescaled Hermite polynomials. The simplification is obtained by first taking the limit $x \to y$, with the help of the de l'Hôpital rule, in the expression (82). Then one sets $y = \lambda_n$, and given that $H_{N+1}(\lambda_n) = 0$, one can obtain the following simplified expression [43, 41]

$$u_{(n)} = \sqrt{\frac{N+1}{N!}} |H_N(\lambda_n)|. \tag{83}$$

Combining Eqs. (80) and (83), it follows that the explicit expression for a generic element of the matrix U_N is given by

$$U_{N_{mn}} = \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_m(\lambda_n)}{|H_N(\lambda_n)|}, \qquad m, n = 0, 1, \cdots, N, \tag{84}$$

so that the generic matrix U_N has the form

$$U_N = \begin{pmatrix} \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_0(\lambda_0)}{|H_N(\lambda_0)|} & \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_0(\lambda_1)}{|H_N(\lambda_1)|} & \cdots & \cdots & \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_0(\lambda_N)}{|H_N(\lambda_N)|} \\ \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_1(\lambda_0)}{|H_N(\lambda_0)|} & \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_1(\lambda_1)}{|H_N(\lambda_1)|} & \cdots & \cdots & \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_1(\lambda_N)}{|H_N(\lambda_N)|} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_N(\lambda_0)}{|H_N(\lambda_0)|} & \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_N(\lambda_1)}{|H_N(\lambda_1)|} & & & \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_N(\lambda_N)}{|H_N(\lambda_N)|} \end{pmatrix}. \tag{85}$$

We are now ready to provide the explicit Hamiltonian structure, in terms of the fluid moments, for an arbitrary member of the class of reduced fluid models under consideration. We formalize this result by means of the following

Proposition 4.1. *For any integer $N \geqslant 1$, the system (63)-(65), with $g_0, g_1, \cdots, g_N \in \mathcal{F}_N$, is a Hamiltonian system with Hamiltonian $H \in \bar{C}^{\infty}(\mathcal{F}_N)$ given by*

$$H(g_0, g_1, \cdots, g_N) = \frac{1}{2}\int_{\mathcal{D}} d^3x \left(\sum_{n=0}^{N} g_n^2 - \sqrt{\frac{2}{\beta_e}}\left(g_0 \phi_{fl}[g_0] - g_1 A_{fl}[g_1]\right) \right), \quad (86)$$

and Poisson bracket given by

$$\{F, K\}_g = \sqrt{\frac{\beta_e}{2}} \sum_{l,m,n=0}^{N} W_{(N)\,l}^{mn} \int_{\mathcal{D}} d^3x \, g_l [F_{g_m}, K_{g_n}]$$
$$- \sum_{m,n=0}^{N} S_{N_{mn}} \int_{\mathcal{D}} d^3x \, F_{g_m} \frac{\partial K_{g_n}}{\partial z}, \quad (87)$$

for $F, K \in \bar{C}^{\infty}(\mathcal{F}_N)$, where

$$W_{(N)\,l}^{mn} = \frac{N!}{N+1} \sum_{i=0}^{N} \frac{H_l(\lambda_i) H_m(\lambda_i) H_n(\lambda_i)}{H_N^2(\lambda_i)\sqrt{l!m!n!}}. \quad (88)$$

In the expression (88), the numbers $\lambda_0, \lambda_1, \cdots, \lambda_N$ are the eigenvalues of the matrix S_N given in Eq. (66), and are known to correspond to the zeros of the Hermite polynomial $H_{N+1}(v)$.

Proof. The system (63)-(65) is equivalent to the system (68)-(70). In particular, given that the orthogonal matrix U_N is invertible with inverse $U_N^{-1} = U_N^T$, one has a linear invertible map $\mathcal{U}_N : \mathcal{G}_N \to \mathcal{F}_N$ defined by

$$\mathcal{U}_N z = U_N z, \quad (89)$$

for $z \in \mathcal{G}_N$, which preserves the properties of a Poisson bracket. Therefore, in order to determine the Hamiltonian structure of the system (63)-(65), it is sufficient to express the Hamiltonian structure of the system (68)-(70), in terms of the variables $g_0, g_1, \cdots, g_N$, making use of the transformation (67) and of its inverse

$$g_m = U_{N_{mn}} G_n, \qquad m = 0, 1, \cdots, N. \quad (90)$$

We proceed with transforming first the Hamiltonian (71). Using the orthogonality of U_N, and Eq. (67), one readily has that

$$\sum_{n=0}^{N} G_n^2 = \sum_{n,i,j=0}^{N} U_{N_{ni}}^T g_i U_{N_{nj}}^T g_j = \sum_{n,i,j=0}^{N} g_j U_{N_{jn}} U_{N_{ni}}^T g_i = \sum_{i=0}^{N} g_i^2, \quad (91)$$

which permits to transform the first term on the right-hand side of Eq. (71). With regard to the remaining terms, they readily follow from the relations $U_{N_{0l}}G_l = g_0$ and $U_{N_{1l}}G_l = g_1$. Thus, one has straightforwardly

$$\mathcal{H}(G_0, G_1, \cdots, G_N) = \frac{1}{2}\int_{\mathcal{D}} d^3x \left(\sum_{n=0}^{N} g_n^2 - \sqrt{\frac{2}{\beta_e}}\left(g_0\phi_{fl}[g_0] - g_1 A_{fl}[g_1]\right)\right)$$
$$= H(g_0, g_1, \cdots, g_N). \tag{92}$$

With regard to the Poisson bracket, in order to transform $\{\, ,\, \}_G$, one needs to transform also the functional derivatives in terms of the variables $g_0, g_1, \cdots, g_N$. This is accomplished using Eqs. (44) and (90), from which one obtains

$$\frac{\delta \bar{F}}{\delta G_i} = U_{N\,ij}^T \frac{\delta F}{\delta g_j}, \qquad i = 0, 1, \cdots, N, \tag{93}$$

for $\bar{F} \in \bar{C}^\infty(\mathcal{G}_N)$ and $F = \bar{F} \circ \mathcal{U}_N^{-1} \in \bar{C}^\infty(\mathcal{F}_N)$.

Using the relations (67) and (93) in the expression (73), for $\bar{F}, \bar{K} \in \bar{C}^\infty(\mathcal{G}_N)$, yields

$$\{\bar{F}, \bar{K}\}_G = \sqrt{\frac{\beta_e}{2}} \sum_{i=0}^{N} \sum_{l,m,n=0}^{N} \frac{1}{U_{N_{0i}}} U_{N_{il}}^T U_{N_{im}}^T U_{N_{in}}^T \int_{\mathcal{D}} d^3x\, g_l [F_{g_m}, K_{g_n}] \tag{94}$$

$$- \sum_{i=0}^{N} \sum_{m,n=0}^{N} \lambda_i U_{N_{im}}^T U_{N_{in}}^T \int_{\mathcal{D}} d^3x\, F_{g_m} \frac{\partial K_{g_n}}{\partial z}$$

$$= \sqrt{\frac{\beta_e}{2}} \sum_{l,m,n=0}^{N} \sum_{i=0}^{N} u_{(i)} \frac{\hat{H}_l(\lambda_i)}{u_{(i)}} \frac{\hat{H}_m(\lambda_i)}{u_{(i)}} \frac{\hat{H}_n(\lambda_i)}{u_{(i)}} \int_{\mathcal{D}} d^3x\, g_l [F_{g_m}, K_{g_n}] \tag{95}$$

$$- \sum_{i=0}^{N} \sum_{m,n=0}^{N} \lambda_i \frac{\hat{H}_m(\lambda_i)}{u_{(i)}} \frac{\hat{H}_n(\lambda_i)}{u_{(i)}} \int_{\mathcal{D}} d^3x\, F_{g_m} \frac{\partial K_{g_n}}{\partial z} \tag{96}$$

$$= \sqrt{\frac{\beta_e}{2}} \sum_{l,m,n=0}^{N} \frac{N!}{N+1} \sum_{i=0}^{N} \frac{H_l(\lambda_i) H_m(\lambda_i) H_n(\lambda_i)}{H_N^2(\lambda_i)\sqrt{l!m!n!}} \int_{\mathcal{D}} d^3x\, g_l [F_{g_m}, K_{g_n}] \tag{97}$$

$$- \frac{N!}{N+1} \sum_{m,n=0}^{N} \sum_{i=0}^{N} \lambda_i \frac{H_m(\lambda_i) H_n(\lambda_i)}{H_N^2(\lambda_i)\sqrt{m!n!}} \int_{\mathcal{D}} d^3x\, F_{g_m} \frac{\partial K_{g_n}}{\partial z}, \tag{98}$$

where $F = \bar{F} \circ \mathcal{U}_N^{-1}$, $K = \bar{K} \circ \mathcal{U}_N^{-1}$, and where in the steps (94)-(98), we made use of the expressions (83) and (84).

Given the definition (88), we see that the expression (97) is already in the desired form. We focus then on the expression (98). Making use of the formula

(77) we obtain

$$
\begin{aligned}
&\frac{N!}{N+1}\sum_{i=0}^{N}\lambda_i\frac{H_m(\lambda_i)H_n(\lambda_i)}{H_N^2(\lambda_i)\sqrt{m!n!}}\\
&=\sum_{m,n=0}^{N}\sum_{i=0}^{N}\sqrt{\frac{N!}{N+1}}\frac{(\sqrt{m+1}\hat{H}_{m+1}(\lambda_i)+\sqrt{m}\hat{H}_{m-1}(\lambda_i))\hat{H}_n(\lambda_i)}{H_N(\lambda_i)}\sqrt{\frac{N!}{N+1}}\frac{\hat{H}_n(\lambda_i)}{H_N(\lambda_i)}\\
&=\sum_{i=0}^{N}(\sqrt{m+1}U_{N_{m+1\,i}}U^T_{N_{in}}+\sqrt{m}U_{N_{m-1\,i}}U^T_{N_{in}})\\
&=(\sqrt{m+1}\delta_{m+1,n}+\sqrt{m}\delta_{m-1,n})=S_{N_{mn}}.
\end{aligned}
\tag{99}
$$

Using the expression (88) and the result (99) in the two final steps of (94)-(98) yields

$$
\begin{aligned}
\{\bar{F},\bar{K}\}_G=&\sqrt{\frac{\beta_e}{2}}\sum_{l,m,n=0}^{N}W^{mn}_{(N)\,l}\int_{\mathcal{D}}d^3x\,g_l[F_{g_m},K_{g_n}]\\
&-\sum_{m,n=0}^{N}S_{N_{mn}}\int_{\mathcal{D}}d^3x\,F_{g_m}\frac{\partial K_{g_n}}{\partial z}\\
=&\{F,K\}_g,
\end{aligned}
\tag{100}
$$

which completes the proof.

□

Remark 4.1. The Poisson bracket (87) generalizes, up to the normalization, various Poisson brackets present in the literature. For instance, for $N=1$, it reduces to the bracket for the electron dynamics of the models of Refs. [24, 25, 44] and to the bracket of the model of Ref. [12] in the cold-ion limit. For $N=2$ one retrieves the bracket for the ion gyrofluid dynamics of the model of Ref. [26], whereas for $N=3$, in the 2D limit, the bracket pertaining to the electron dynamics with heat flux of Ref. [27] is obtained.

Remark 4.2. Comparing Eq. (87) with Eq. (73), i.e. the expressions of the Poisson bracket in terms of the fluid moments $g_0, g_1, \cdots, g_N$ and in terms of the variables $G_0, G_1, \cdots, G_N$, respectively, it emerges that the Poisson bracket takes a much simpler form in terms of the latter variables. In particular, when considering the perpendicular component $\{F,K\}_{G_\perp}$, one sees that this is the direct sum of independent Poisson brackets of the form

$$c_i \int_{\mathcal{D}} d^3x\, G_i[F_{G_i}, K_{G_i}], \tag{101}$$

with constant coefficients $c_0, c_1, \cdots, c_N$. The direct sum is one of the ways in which one can build a Lie-Poisson bracket by extension [20]. As a consequence, the proof of the Jacobi indentity is much easier when the bracket is expressed in terms of the variables $G_0, G_1, \cdots, G_N$. This represents one of the main advantages of introducing such variables.

5. Comparison with the approach based on a truncated series

In this Section we compare the Hamiltonian (86) and the Poisson bracket (87) with the functionals and the bilinear form, respectively, that one derives from the parent drift-kinetic Hamiltonian (49) and Poisson bracket (50), upon replacing, as dynamical variable, the generalized perturbed distribution function g with its truncated series retaining only the first $N+1$ Hermite moments. In order to carry out the comparison, we first present, with the next Proposition 5.2, a reformulation of the Poisson bracket (87). The proof of Proposition 5.2 is preceded by the following

Lemma 5.1. *For every integer $N \geqslant 1$, the coefficients $W^{mn}_{(N)\,l}$ in Eq. (88) possess the following properties:*

(a) $W^{mn}_{(N)\,l} = 0$ *if $l+m+n$ is an odd number,*

(b) *For fixed integers l, m, n, one has $W^{\sigma(m)\sigma(n)}_{N\,\sigma(l)} = W^{mn}_{(N)\,l}$, where $\sigma : \{l,m,n\} \to \{l,m,n\}$ is a permutation of the integers l,m,n.*

The proof of Lemma 5.1 is provided in AppendixA.

Remark 5.1. We observe that the property $W^{mn}_{(N)\,l} = W^{nm}_{(N)\,l}$, following from Lemma 5.1 (b), is required by the antisymmetry of the Poisson bracket $\{\,,\}_g$ [20].

We now proceed with re-expressing the Poisson bracket $\{\,,\,\}_g$ in a way that will facilitate its comparison with the bilinear form obtained from the truncated series, which will be derived in Sec. 5.1. In order to formulate the corresponding Proposition, it is convenient to the define, for a given integer $N \geqslant 1$, the following sets A_N and B_N:

$$\begin{aligned} A_N = \{&(l,m,n) \in \mathbb{Z}^3_{\geqslant 0} \,|\, l,m,n \leqslant N \,,\, l+m+n \text{ is even}, \\ &m+n \geqslant l \,,\, n+l \geqslant m \,,\, l+m \geqslant n\}, \end{aligned} \tag{102}$$

$$\begin{aligned} B_N = \{&(l,m,n) \in \mathbb{Z}^3_{\geqslant 0} \,|\, l,m,n \leqslant N \,,\, l+m+n \text{ is even}, \\ &m+n > N+1 \,,\, n+l > N+1 \,,\, l+m > N+1\}. \end{aligned} \tag{103}$$

The set B_N is thus a subset of A_N.

Also, for two given positive integers m and n, such that $m+n > N+1$, we introduce the number $r_{N_{mn}}$ defined by

$$r_{N_{mn}} = \min(R_{N_{mn}}), \tag{104}$$

where

$$R_{N_{mn}} = \{r \in \mathbb{Z}_{>0} : (m+n-N-1)/2 \leqslant r \leqslant \min(m,n)\}. \tag{105}$$

We can now formulate the following

Proposition 5.2. *Given two functionals $F, K \in \bar{C}^\infty(\mathcal{F}_N)$, the Poisson bracket $\{\,,\,\}_g$ can be expressed in the following way:*

$$\{F,K\}_g = \{F,K\}_{g_\perp} + \{F,K\}_{g_\parallel}. \tag{106}$$

In the expression (106) one has

$$\{F,K\}_{g_\perp} = \sqrt{\frac{\beta_e}{2}} \sum_{l,m,n=0}^{N} W^{mn}_{(N)\,l} \int_{\mathcal{D}} d^3x \, g_l [F_{g_m}, K_{g_n}], \tag{107}$$

where

$$W^{mn}_{(N)\,l}=\begin{cases}\dfrac{\sqrt{l!m!n!}}{\left(\frac{l+m-n}{2}\right)!\left(\frac{n+l-m}{2}\right)!\left(\frac{m+n-l}{2}\right)!}, & \textit{if } (l,m,n)\in A_N\backslash B_N,\\[2em] \dfrac{N!}{N+1}\displaystyle\sum_{i=0}^{N}\frac{H_l(\lambda_i)}{H_N^2(\lambda_i)}\sum_{r=0}^{r_{N_{mn}}-1}\frac{m!}{(m-r)!}\frac{n!}{(n-r)!r!}\frac{H_{m+n-2r}(\lambda_i)}{\sqrt{l!m!n!}} & \\ +\dfrac{\sqrt{l!m!n!}}{\left(\frac{l+m-n}{2}\right)!\left(\frac{n+l-m}{2}\right)!\left(\frac{m+n-l}{2}\right)!}, & \textit{if } (l,m,n)\in B_N,\\[2em] 0 & \textit{otherwise}\end{cases} \tag{108}$$

and

$$\{F,K\}_{g_\parallel}=-\sum_{m,n=0}^{N}S_{N_{mn}}\int_{\mathcal{D}}d^3x\,F_{g_m}\frac{\partial K_{g_n}}{\partial z}. \tag{109}$$

The proof of Proposition 5.2 is provided in AppendixB.

5.1. Poisson bracket $\{\,,\,\}_g$ vs. bilinear structure obtained from truncated series

The closed fluid system (63)-(65) can be obtained from the drift-kinetic system (2)-(4), assuming that g, at any time t, is a truncated Hermite series, retaining only the moments from order 0 up to order N. Therefore it is natural to ask whether the Hamiltonian and the Poisson bracket of the fluid model (63)-(65) can be obtained from those of the drift-kinetic model, by replacing g, in the latter, with its truncated series, or, more precisely, by restricting the Poisson algebra of observables, to functionals of Hermite series truncated at the moment of order N, and then expressing the outcome of the resulting restricted algebra in terms of the fluid moment variables. A delicate point in this operation concerns the Poisson bracket. Indeed, in order for the set of functionals of the truncated Hermite series to be closed under the restricted bilinear algebra operation (descending from the drift-kinetic Poisson bracket), one has to restrict the Poisson operator of the drift-kinetic bracket, taking its composition with the projector onto the subspace of truncated Hermite series. Because this operation is not invertible, the resulting bilinear form can fail to satisfy the Jacobi identity and therefore not be a Poisson bracket.

In order to carry out this analysis, let us first introduce the operator $\mathcal{P}_N : \mathcal{G} \to \Gamma_N$ defined by

$$\mathcal{P}_N(g) = \sum_{n=0}^{N} g_n \frac{H_n}{\sqrt{n!}} F_{eq}. \tag{110}$$

The operator $\mathcal{P}_N$ projects a generalized perturbed distribution function g onto the subspace

$$\Gamma_N = \{g \in \mathcal{G} |\, g_i = 0 \quad \forall i \geqslant N+1\}, \tag{111}$$

whose elements are the truncated Hermite series. Each $\tilde{g} \in \Gamma_N$ can be written as

$$\tilde{g}(\mathbf{x}, v) = \sum_{n=0}^{N} g_n(\mathbf{x}) \frac{H_n(v)}{\sqrt{n!}} F_{eq}(v), \tag{112}$$

with $(g_0, g_1, \cdots, g_N) \in \mathcal{F}_N$ satisfying

$$g_n = \frac{1}{\sqrt{n!}} \int_{-\infty}^{+\infty} dv\, H_n\, \tilde{g}, \qquad n = 0, 1, \cdots, N. \tag{113}$$

We want to identify the restriction of the Poisson algebra of observables of the drift-kinetic system, consisting of $\bar{C}^\infty(\mathcal{G})$, to the algebra $\bar{C}^\infty(\Gamma_N)$ of the truncated Hermite series, with the corresponding restricted bilinear algebra operation. In particular, we want to determine the expression of the restricted Hamiltonian functional $\mathbb{H} : \mathcal{F}_N \to \mathbb{R}$ defined by

$$\mathbb{H}(g_0, g_1, \cdots, g_N) = H_{dk} \circ \mathcal{P}_N(g) = H_{dk}(\tilde{g}), \tag{114}$$

for $\tilde{g} \in \Gamma_N$ given by Eq. (112), and of the bilinear operator $[\![\, ,\,]\!] : \bar{C}^\infty(\mathcal{F}_N) \times \bar{C}^\infty(\mathcal{F}_N) \to \bar{C}^\infty(\mathcal{F}_N)$ given by

$$\begin{aligned}
[\![F, K]\!] &= \left\langle \frac{\delta \bar{F} \circ \mathcal{P}_N(g)}{\delta g}, J_{dk} \circ \mathcal{P}_N(g) \frac{\delta \bar{K} \circ \mathcal{P}_N(g)}{\delta g} \right\rangle \\
&= \left\langle \frac{\delta \bar{F}}{\delta \tilde{g}}, J_{dk}(\tilde{g}) \frac{\delta \bar{K}}{\delta \tilde{g}} \right\rangle
\end{aligned} \tag{115}$$

where $\bar{F}, \bar{K} \in \bar{C}^\infty(\Gamma_N)$ and $F(g_0, g_1, \cdots, g_N) = \bar{F} \circ \mathcal{P}_N(g) = \bar{F}(\tilde{g})$, $K(g_0, g_1, \cdots, g_N) = \bar{K} \circ \mathcal{P}_N(g) = \bar{K}(\tilde{g})$, with $\tilde{g} \in \Gamma_N$.

In Eqs. (114) and (115), the functional H_{dk}, the Poisson bracket $\{\, ,\, \}_{dk}$ and the Poisson operator J_{dk} are those of the drift-kinetic model given by Eqs. (49), (50) and (54), respectively. In Eq. (115), as above mentioned, we are considering a bilinear form involving the composition of the Poisson operator J_{dk}, with the projector $\mathcal{P}_N$. The composition of these two operators, in general, is not a Poisson operator. Therefore, the bilinear form (115) is not guaranteed to be a Poisson bracket.

After having defined the restricted Hamiltonian functional (114) and bilinear algebra operation (115), we intend to find their explicit expressions in terms of the fluid moments $g_0, g_1, \cdots, g_N$, in order to compare them with the Hamiltonian (86) and the Poisson bracket (106) of the fluid model.

The wanted expressions for $\mathbb{H}$ and $[\![\,,\,]\!]$ are provided by the next two Propositions.

Proposition 5.3. *The functional $\mathbb{H}$, defined by the relation (114), coincides with the Hamiltonian functional H of the fluid model given in Eq. (86) of Proposition 4.1.*

Proof. From Eqs. (114), (49) and (112) one has

$$\mathbb{H}(g_0, g_1, \cdots, g_N) = H_{dk}(\tilde{g}) = \frac{1}{2}\int_{\mathcal{D}} d^3x \int_{-\infty}^{+\infty} dv \left(\sum_{n,n'=0}^{N} g_n \frac{H_n}{\sqrt{n!}} g_{n'} \frac{H_{n'}}{\sqrt{n'!}} F_{eq} \right. \tag{116}$$

$$\left. -\sqrt{\frac{2}{\beta_e}} \sum_{n=0}^{N} g_n \frac{H_n}{\sqrt{n!}} F_{eq} \left(\phi_{dk}\left[\sum_{n'=0}^{N} g_{n'} \frac{H_{n'}}{\sqrt{n'!}} F_{eq} \right] - v A_{dk} \left[\sum_{n'=0}^{N} g_{n'} \frac{H_{n'}}{\sqrt{n'!}} F_{eq} \right] \right) \right) \tag{117}$$

$$= \frac{1}{2}\int_{\mathcal{D}} d^3x \left(\sum_{n=0}^{N} g_n^2 - \sqrt{\frac{2}{\beta_e}} \left(g_0 \phi_{fl}[g_0] - g_1 A_{fl}[g_1] \right) \right) = H(g_0, g_1, \cdots, g_N). \tag{118}$$

To go from the expressions (116)-(117) to the expressions (118) we made use of the orthogonality relation (20) for Hermite polynomials and of the relations (41), permitting to express electromagnetic potentials in terms of fluid moments.

□

Proposition 5.4. *Given two functionals $F, K \in \bar{C}^{\infty}(\mathcal{F}_N)$, the bilinear operator*

$[\![\,,\,]\!]$, *defined by the relation (115), can be written as*

$$[\![F,K]\!] = [\![F,K]\!]_\perp + [\![F,K]\!]_\parallel, \tag{119}$$

where

$$[\![F,K]\!]_\perp = \sqrt{\frac{\beta_e}{2}} \sum_{l,m,n=0}^{N} \mathbb{W}^{mn}_{(N)\,l} \int_{\mathcal{D}} d^3x\, g_l[F_{g_m}, K_{g_n}], \tag{120}$$

with

$$\mathbb{W}^{mn}_{(N)\,l} = \begin{cases} \dfrac{\sqrt{l!m!n!}}{\left(\frac{l+m-n}{2}\right)!\left(\frac{n+l-m}{2}\right)!\left(\frac{m+n-l}{2}\right)!}, & \textit{if } (l,m,n) \in A_N, \\ \\ 0 \qquad \textit{otherwise} \end{cases} \tag{121}$$

and

$$[\![F,K]\!]_\parallel = -\sum_{m,n=0}^{N} S_{N_{mn}} \int_{\mathcal{D}} d^3x\, F_{g_m} \frac{\partial K_{g_n}}{\partial z}. \tag{122}$$

Proof. From Eq. (115), using Eqs. (50) and (112), we obtain

$$\begin{aligned} [\![F,K]\!] &= \left\langle \frac{\delta \bar{F}}{\delta \tilde{g}}, J_{dk}(\tilde{g}) \frac{\delta \bar{K}}{\delta \tilde{g}} \right\rangle \\ &= \int_{\mathcal{D}} d^3x \int_{-\infty}^{+\infty} dv \left(\sqrt{\frac{\beta_e}{2}} \tilde{g}[\bar{F}_{\tilde{g}}, \bar{K}_{\tilde{g}}] - vF_{eq}\bar{F}_{\tilde{g}} \frac{\partial \bar{K}_{\tilde{g}}}{\partial z} \right). \end{aligned} \tag{123}$$

From the definitions of functional derivatives (42) and (44), using also the relations (112) and (113), one obtains the following chain rule for functional derivatives:

$$\frac{\delta \bar{F}}{\delta \tilde{g}}(\mathbf{x}, v) = \sum_{n=0}^{N} \frac{H_n(v)}{\sqrt{n!}} \frac{\delta F}{\delta g_n}(\mathbf{x}), \tag{124}$$

for $\bar{F}(\tilde{g}) = F(g_0, g_1, \cdots, g_N)$. Inserting the expressions (112) and (124) into Eq. (123) yields

$$\begin{aligned} [\![F,K]\!] = \int_{\mathcal{D}} d^3x \int_{-\infty}^{+\infty} dv &\left(\sqrt{\frac{\beta_e}{2}} \sum_{l,m,n=0}^{N} F_{eq} \frac{H_l H_m H_n}{\sqrt{l!m!n!}} g_l[F_{g_m}, K_{g_n}] \right. \\ &\left. - \sum_{m,n=0}^{N} F_{eq} \frac{H_1 H_m H_n}{\sqrt{m!n!}} F_{g_m} \frac{\partial K_{g_n}}{\partial z} \right), \end{aligned} \tag{125}$$

where we also made use of the relation $v = H_1(v)$.

At this point, we can apply to the expression (125) the following identity for Hermite polynomials [45]:

$$\int_{-\infty}^{+\infty} dv\, F_{eq}(v) \frac{H_l(v)H_m(v)H_n(v)}{\sqrt{l!m!n!}} \tag{126}$$

$$= \begin{cases} \dfrac{\sqrt{l!m!n!}}{\left(\frac{l+m-n}{2}\right)!\left(\frac{n+l-m}{2}\right)!\left(\frac{m+n-l}{2}\right)!}, & \text{if } l+m+n \text{ is even and } l+m \geqslant n,\ m+n \geqslant l,\ n+l \geqslant m \\ \\ 0 & \text{otherwise.} \end{cases} \tag{127}$$

In particular, with regard to the expression in the second line of Eq. (125), we have that

$$\int_{-\infty}^{+\infty} dv\, F_{eq} \frac{H_1 H_m H_n}{\sqrt{m!n!}} = \frac{\sqrt{m!n!}}{\left(\frac{1+m-n}{2}\right)!\left(\frac{m+n-1}{2}\right)!\left(\frac{n+1-m}{2}\right)!} \tag{128}$$

when $m+n+1$ is even and the three conditions

$$1+m \geqslant n, \qquad m+n \geqslant 1, \qquad n+1 \geqslant m, \tag{129}$$

are satisfied. Otherwise,

$$\int_{-\infty}^{+\infty} dv\, F_{eq} \frac{H_1 H_m H_n}{\sqrt{m!n!}} = 0. \tag{130}$$

The conditions (129), together with the constraint for $m+n+1$ of being even, imply

$$m-1 \leqslant n \leqslant m+1. \tag{131}$$

From the relation (131) it follows that the integral (128) can be non-zero only if $n = m-1$, $n = m$ or $n = m+1$. However, the case $n = m$ implies that $m+n+1 = 2m+1$ is odd. Therefore, also in this case the integral is zero. Using Eq. (128), it follows that

$$\int_{-\infty}^{+\infty} dv\, F_{eq} \frac{H_1 H_m H_n}{\sqrt{m!n!}} = \sqrt{m+1}\delta_{m+1,n} + \sqrt{m}\delta_{m-1,n} = S_{N_{mn}}. \tag{132}$$

Using Eqs. (126) (recalling that l, m and n go from 0 to N) and (132) into Eq.

(125), we can write

$$
\llbracket F, K \rrbracket = \sqrt{\frac{\beta_e}{2}} \sum_{l,m,n=0}^{N} \mathbb{W}_{(N)\,l}^{mn} \int_{\mathcal{D}} d^3x\, g_l [F_{g_m}, K_{g_n}]
$$

$$
- \sum_{m,n=0}^{N} S_{N_{mn}} \int_{\mathcal{D}} d^3x\, F_{g_m} \frac{\partial K_{g_n}}{\partial z}, \tag{133}
$$

where

$$
\mathbb{W}_{(N)\,l}^{mn} = \begin{cases} \dfrac{\sqrt{l!m!n!}}{\left(\frac{l+m-n}{2}\right)!\left(\frac{n+l-m}{2}\right)!\left(\frac{m+n-l}{2}\right)!}, & \text{if } (l,m,n) \in A_N, \\ \\ 0 \qquad \text{otherwise.} \end{cases} \tag{134}
$$

Therefore we can conclude

$$
\llbracket F, K \rrbracket = \llbracket F, K \rrbracket_{\perp} + \llbracket F, K \rrbracket_{\|}. \tag{135}
$$

□

From Proposition 5.3 it follows that, for any N, the Hamiltonian of the fluid model can be derived from the Hamiltonian of the drift-kinetic model, by replacing, in the latter, the generalized perturbed distribution function with its truncated series of order N. On the other hand, Proposition 5.4 shows that the Poisson bracket of the fluid model cannot be obtained from the Poisson bracket of the drift-kinetic model, by considering a Poisson operator depending only on truncated Hermite series, and by restricting the set of observables to $\bar{C}^{\infty}(\mathcal{F}_N)$. More in detail, by comparing Proposition 5.2 with Proposition 5.4 one sees that

$$
\{\, , \,\}_{g_{\|}} = \llbracket\, , \,\rrbracket_{\|}. \tag{136}
$$

However, the difference arises in the "perpendicular' components $\{\, , \,\}_{g_{\perp}}$ and $\llbracket\, , \,\rrbracket_{\perp}$, characterized by the coefficients $W_{(N)\,l}^{mn}$ and $\mathbb{W}_{(N)\,l}^{mn}$. More specifically, one sees that in general $W_{(N)\,l}^{mn} = \mathbb{W}_{(N)\,l}^{mn}$ except when $(l,m,n) \in B_N$, i.e. when l, m and n are all sufficiently 'large'. For most of the values of l, m and n, the coefficients of the bilinear form, obtained from the truncated series, and the

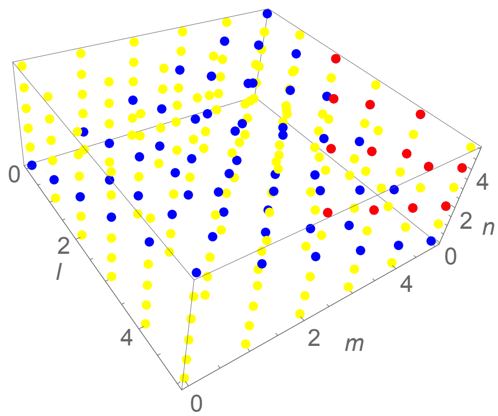

Figure 1: Plot illustrating, in the lmn space, the points where $W^{mn}_{(N)\,l} = \mathbb{W}^{mn}_{(N)\,l} = 0$, the points of the set $A_N \backslash B_N$ and the points of the set B_N, for the case $N = 5$. The yellow points are those for which $W^{mn}_{(N)\,l} = \mathbb{W}^{mn}_{(N)\,l} = 0$. The blue points are the points of the set $A_N \backslash B_N$. At such points one has $W^{mn}_{(N)\,l} = \mathbb{W}^{mn}_{(N)\,l}$. The red color indicate the points of the set B_N. At such points $W^{mn}_{(N)\,l} \neq \mathbb{W}^{mn}_{(N)\,l}$. It is at such points that the coefficients of $\{\,,\,\}_{g_\perp}$ differ from those of $[\![\,,\,]\!]_\perp$.

coefficients of the Poisson bracket, are nevertheless identical. It is in this sense, that we previously stated that the bilinear form $[\![\,,\,]\!]$ and the Poisson bracket $\{\,,\,\}_g$ are "very similar". Note also that the expression for the coefficients $\mathbb{W}^{mn}_{(N)\,l}$, given by Eq. (121), does not depend on N, whereas this is the case for $W^{mn}_{(N)\,l}$ when $(l, m, n) \in B_N$, as shown by Eq. (108).

The distribution, in the lmn space, of the points where $W^{mn}_{(N)\,l} = \mathbb{W}^{mn}_{(N)\,l} = 0$, of the points of $A_N \backslash B_N$ (where $W^{mn}_{(N)\,l} = \mathbb{W}^{mn}_{(N)\,l}$) and of the points of B_N (where $W^{mn}_{(N)\,l} \neq \mathbb{W}^{mn}_{(N)\,l}$) is illustrated, for the case $N = 5$, in Fig. 1.

6. Example : a three-moment model

In order to exemplify the results presented in Secs. 4 and 5, we treat here in detail the case $N = 2$. In this case, the resulting fluid model evolves the three moments g_0, g_1 and g_2, which, as anticipated in 3.1, are proportional to the fluctuations of electron gyrocenter density, parallel canonical momentum and parallel temperature, respectively. The closure adopted in this case is $g_3 = 0$, which amounts to set the parallel heat flux fluctuations of the electron gyrocenters equal to zero.

The matrix S_2 reads

$$S_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \tag{137}$$

and its eigenvalues, corresponding to the zeros of $H_3(x) = x^3 - 3x$, are

$$(\lambda_0, \lambda_1, \lambda_2) = (-\sqrt{3}, 0, \sqrt{3}). \tag{138}$$

From Eq. (86) of Proposition 4.1, we have that the Hamiltonian of the fluid model is given by

$$H(g_0, g_1, g_2) = \frac{1}{2}\int_{\mathcal{D}} d^3x \left(g_0^2 + g_1^2 + g_2^2 - \sqrt{\frac{2}{\beta_e}}\,(g_0\phi_{fl}[g_0] - g_1 A_{fl}[g_1]) \right). \tag{139}$$

For the Poisson bracket, we consider the expression following from Eq. (106) of Proposition 5.2. For two functionals $F, K \in \bar{C}^\infty(\mathcal{F}_2)$ the Poisson bracket can thus be written as

$$\begin{aligned} \{F, K\}_g = {} & \sqrt{\frac{\beta_e}{2}} \sum_{l,m,n=0}^{2} W_{(2)\,l}^{mn} \int_{\mathcal{D}} d^3x\, g_l [F_{g_m}, K_{g_n}] \\ & - \sum_{m,n=0}^{2} S_{2_{mn}} \int_{\mathcal{D}} d^3x\, F_{g_m} \frac{\partial K_{g_n}}{\partial z}, \end{aligned} \tag{140}$$

where

$$W_{(2)\,l}^{mn} = 0, \qquad \text{for } 0 \leqslant l, m, n \leqslant 2 \quad \text{except for}$$

$$\begin{aligned} & W_{(2)\,0}^{00} = 1, \\ & W_{(2)\,0}^{11} = W_{(2)\,1}^{10} = W_{(2)\,1}^{01} = 1, \\ & W_{(2)\,0}^{22} = W_{(2)\,2}^{20} = W_{(2)\,2}^{02} = 1, \\ & W_{(2)\,2}^{11} = W_{(2)\,1}^{12} = W_{(2)\,1}^{21} = \sqrt{2}, \\ & W_{(2)\,2}^{22} = \frac{1}{\sqrt{2}}. \end{aligned} \tag{141}$$

In a less compact form, the Poisson bracket (140) can be written as

$$
\begin{aligned}
\{F, K\}_g = & \int_{\mathcal{D}} d^3x \left(\sqrt{\frac{\beta_e}{2}} (g_0([F_{g_0}, K_{g_0}] + [F_{g_1}, K_{g_1}] + [F_{g_2}, K_{g_2}]) \right. \\
& + g_1([F_{g_0}, K_{g_1}] + [F_{g_1}, K_{g_0}] + \sqrt{2}[F_{g_1}, K_{g_2}] + \sqrt{2}[F_{g_2}, K_{g_1}]) \\
& + g_2([F_{g_0}, K_{g_2}] + [F_{g_2}, K_{g_0}] + \sqrt{2}[F_{g_1}, K_{g_1}] + \frac{1}{\sqrt{2}}[F_{g_2}, K_{g_2}])) \qquad (142) \\
& \left. - F_{g_0} \frac{\partial K_{g_1}}{\partial z} - F_{g_1} \frac{\partial K_{g_0}}{\partial z} - \sqrt{2} F_{g_1} \frac{\partial K_{g_2}}{\partial z} - \sqrt{2} F_{g_2} \frac{\partial K_{g_1}}{\partial z} \right).
\end{aligned}
$$

The Hamiltonian (139) and the Poisson bracket (142) yield the following equations of motion:

$$
\begin{aligned}
& \frac{\partial g_0}{\partial t} + [\phi, g_0] - [A, g_1] + \frac{\partial g_1}{\partial z} + \sqrt{\frac{2}{\beta_e}} \frac{\partial A}{\partial z} = 0, \\
& \frac{\partial g_1}{\partial t} + [\phi, g_1] - [A, g_0] - \sqrt{2}[A, g_2] + \frac{\partial g_0}{\partial z} + \sqrt{2} \frac{\partial g_2}{\partial z} - \sqrt{\frac{2}{\beta_e}} \frac{\partial \phi}{\partial z} = 0, \qquad (143) \\
& \frac{\partial g_2}{\partial t} + [\phi, g_2] - \sqrt{2}[A, g_1] + \sqrt{2} \frac{\partial g_1}{\partial z} + \sqrt{2} \sqrt{\frac{2}{\beta_e}} \frac{\partial A}{\partial z} = 0,
\end{aligned}
$$

where $\phi = \phi_{fl}[g_0]$ and $A = A_{fl}[g_1]$. The system (143) coincides namely with the fluid system (63) in the case $N = 2$, obtained by truncating the hierarchy of equations (56)-(59) with $g_3 = 0$.

Also, from Eq. (85) one obtains

$$
U_2 = \begin{pmatrix} \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \qquad (144)
$$

(note a difference with respect to Eq. (87) of Ref. [33], due to a different numbering of the eigenvalues and a different choice in the normalization constant). From the matrix U_2 one can, by means of Eq. (67), obtain the expressions for the variables $(G_0, G_1, G_2) \in \mathcal{G}_2$, which read

$$
\begin{aligned}
G_0 &= \frac{g_0}{\sqrt{6}} - \frac{g_1}{\sqrt{2}} + \frac{g_2}{\sqrt{3}}, \\
G_1 &= \sqrt{\frac{2}{3}} g_0 - \frac{g_2}{\sqrt{3}}, \qquad (145) \\
G_2 &= \frac{g_0}{\sqrt{6}} + \frac{g_1}{\sqrt{2}} + \frac{g_2}{\sqrt{3}}.
\end{aligned}
$$

In terms of these variables, following Eqs. (68)-(70), the system (143) can be rewritten as

$$\begin{aligned}
&\frac{\partial G_0}{\partial t} + [\phi + \sqrt{3}A, G_0] - \sqrt{3}\frac{\partial G_0}{\partial z} + \sqrt{\frac{2}{\beta_e}}\frac{\partial}{\partial z}\left(\sqrt{\frac{3}{2}}A + \frac{\phi}{\sqrt{2}}\right) = 0,\\
&\frac{\partial G_1}{\partial t} + [\phi, G_1] = 0, \qquad (146)\\
&\frac{\partial G_2}{\partial t} + [\phi - \sqrt{3}A, G_2] + \sqrt{3}\frac{\partial G_2}{\partial z} + \sqrt{\frac{2}{\beta_e}}\frac{\partial}{\partial z}\left(\sqrt{\frac{3}{2}}A - \frac{\phi}{\sqrt{2}}\right) = 0,
\end{aligned}$$

where $\phi = \phi_{fl}[U_{2_{0l}} G_l]$ and $A = A_{fl}[U_{2_{1l}} G_l]$.

The formulation (146) puts in evidence the existence of a Lagrangian invariant, corresponding to G_1, which is simply transported by the velocity field

$$\mathbf{v}_{\mathbf{E}\times\mathbf{B}} = \hat{z} \times \nabla\phi, \qquad (147)$$

which identifies with the so-called $\mathbf{E} \times \mathbf{B}$ velocity field. We note that, on the basis of our definitions of the fluid moments, G_1 is proportional to $n_e - T_{\|e}/2$, where n_e and $T_{\|e}$ are normalized fluctuations of electron gyrocenter density and parallel temperature, respectively. We observe that the relation $T_{\|e} = 2n_e$ (implying $G_1 = 0$) expresses an adiabatic law for the parallel electron temperature. Therefore, if $T_{\|e} = 2n_e$ at $t = 0$, $G_1 = 0$ is a solution of Eq. (146), which expresses the adiabatic relation between parallel electron temperature and density at all times. If $T_{\|e} \neq 2n_e$ at $t = 0$, on the other hand, such initial departure from adiabaticity is conserved along the flow of the $\mathbf{E} \times \mathbf{B}$ velocity at all times. From Eqs. (146) it can also be seen that, in the 2D limit in which the fields G_0, G_1 and G_2 do not depend on the z coordinate, all these three fields become Lagrangian invariants, as already pointed out in Refs. [32, 33].

We now apply, to the case $N = 2$, the approach based on the truncated series. From Proposition 5.3 we obtain that the truncated Hermite series $\tilde{g} \in \Gamma_2$, given by

$$\tilde{g}(\mathbf{x}, v) = \sum_{n=0}^{2} g_n(\mathbf{x}) \frac{H_n(v)}{\sqrt{n!}} F_{eq}(v), \qquad (148)$$

when inserted into the drift-kinetic Hamiltonian H_{dk}, yields the functional $\mathbb{H}(g_0, g_1, g_2) = H(g_0, g_1, g_2)$, where H is the fluid Hamiltonian (139). On the other hand, the expression of the bilinear form $[\![\,,\,]\!]$, for two functionals $F, K \in \bar{C}^{\infty}(\mathcal{F}_2)$, follows from Proposition 5.4 and is given by

$$[\![F, K]\!] = \sqrt{\frac{\beta_e}{2}} \sum_{l,m,n=0}^{2} \mathbb{W}_{(2)\,l}^{mn} \int_{\mathcal{D}} d^3x\, g_l [F_{g_m}, K_{g_n}] \\ - \sum_{m,n=0}^{2} S_{2_{mn}} \int_{\mathcal{D}} d^3x\, F_{g_m} \frac{\partial K_{g_n}}{\partial z}, \tag{149}$$

where

$$\mathbb{W}_{(2)\,l}^{mn} = 0, \qquad \text{for } 0 \leqslant l, m, n \leqslant 2 \quad \text{except for}$$

$$\begin{aligned} &\mathbb{W}_{(2)\,0}^{00} = 1, \\ &\mathbb{W}_{(2)\,0}^{11} = \mathbb{W}_{(2)\,1}^{10} = \mathbb{W}_{(2)\,1}^{01} = 1, \\ &\mathbb{W}_{(2)\,0}^{22} = \mathbb{W}_{(2)\,2}^{20} = \mathbb{W}_{(2)\,2}^{02} = 1, \\ &\mathbb{W}_{(2)\,2}^{11} = \mathbb{W}_{(2)\,1}^{12} = \mathbb{W}_{(2)\,1}^{21} = \sqrt{2}, \\ &\mathbb{W}_{(2)\,2}^{22} = 2\sqrt{2}. \end{aligned} \tag{150}$$

Comparing Eq. (142) with Eq. (149) and Eq. (141) with Eq. (150), it emerges that the bilinear form $[\![\,,\,]\!]$ differs from the Poisson bracket $\{\,,\,\}_g$ only by the coefficient $\mathbb{W}_{(2)\,2}^{22} = 2\sqrt{2}$, which is not equal to $W_{(2)\,2}^{22} = 1/\sqrt{2}$. Indeed, the set B_2 is given by $B_2 = \{(2,2,2)\}$ and, from Eq. (108) of Proposition 5.2, it follows that only the expression for $W_{(2)\,2}^{22} = 1/\sqrt{2}$ differs from the expression of the corresponding coefficient of $[\![\,,\,]\!]$.

Remark 6.1. It turns out that the bilinear form (149) is antisymmetric, satisfies the Leibniz rule but it does not satisfy the Jacobi identity. Therefore, the form (149) *is not a Poisson bracket*. In order to see this, we first remark that the Jacobi identity for the bilinear form $[\![\,,\,]\!]$ can be written as

$$[\![[\![F, K]\!]_{\perp}, E]\!]_{\perp} + [\![[\![F, K]\!]_{\perp}, E]\!]_{\parallel} + [\![[\![F, K]\!]_{\parallel}, E]\!]_{\perp} + [\![[\![F, K]\!]_{\parallel}, E]\!]_{\parallel} + \circlearrowleft = 0, \tag{151}$$

where the symbol $\circlearrowleft$ indicates the additional terms obtained by cyclic permutation of F, K and E. The identity (151) must be valid for all functionals $F, K, E \in \bar{C}^{\infty}(\mathcal{F}_2)$.

In order to investigate the relation (151) we can make use of a result presented in Sec. 3.2 of Ref. [23]. According to such result, one has that

$$[\![[\![F,K]\!]_{\|},E]\!]_{\perp}+[\![[\![F,K]\!]_{\|},E]\!]_{\|}+\circlearrowleft=0, \tag{152}$$

which follows from the fact that $[\![\,,\,]\!]_{\|}$ is itself a Poisson bracket with constant Poisson operator. With regard to the expression

$$[\![[\![F,K]\!]_{\perp},E]\!]_{\|}+\circlearrowleft, \tag{153}$$

it vanishes for all $F,K,E\in\bar{C}^{\infty}(\mathcal{F}_2)$ if and only if the condition

$$S_{2_{in}}\mathbb{W}^{jm}_{(2)\,i}=S_{2_{ij}}\mathbb{W}^{mn}_{(2)\,i}=S_{2_{im}}\mathbb{W}^{nj}_{(2)\,i},\qquad \text{for } 0\leqslant j,m,n\leqslant 2, \tag{154}$$

(where the sum over the repeated index i is understood) is satisfied. However, if we consider the case $j=2,m=2,n=1$, from Eqs. (137) and (150), we obtain

$$S_{2_{i1}}\mathbb{W}^{22}_{(2)\,i}=5,\qquad S_{2_{i2}}\mathbb{W}^{21}_{(2)\,i}=2. \tag{155}$$

Therefore $S_{2_{i1}}\mathbb{W}^{22}_{(2)\,i}\neq S_{2_{i2}}\mathbb{W}^{21}_{(2)\,i}$ and the condition (154) is not satisfied. Thus, the expression (153) does not vanish for all $F,K,E\in\bar{C}^{\infty}(\mathcal{F}_2)$. Similarly, one can show that the

$$[\![[\![F,K]\!]_{\perp},E]\!]_{\perp}+\circlearrowleft=0, \tag{156}$$

does not hold for all $F,K,E\in\bar{C}^{\infty}(\mathcal{F}_2)$. An effective way to see this is to take advantage from a result in Sec. 2.3 of Ref. [20] (actually, the result of Ref. [20] refers to a purely 2D domain, but this difference is irrelevant for the present purpose). According to this result, for a given N, antisymmetric bilinear operations of the form (120) satisfy the Jacobi identity if and only if all the $N+1$ matrices $\mathbb{W}^{(k)}$, whose elements of row i and column j are defined by

$$\mathbb{W}^{(k)}_{ij}=\mathbb{W}^{jk}_{(N)\,i},\qquad i,j,k=0,\cdots,N, \tag{157}$$

pairwise commute.

In the present case $N=2$, from Eq. (150), one obtains that the three matrices $\mathbb{W}^{(0)}, \mathbb{W}^{(1)}$ and $\mathbb{W}^{(2)}$ are given by

$$\mathbb{W}^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbb{W}^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \qquad \mathbb{W}^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 2\sqrt{2} \end{pmatrix}. \tag{158}$$

Clearly, both $\mathbb{W}^{(1)}$ and $\mathbb{W}^{(2)}$ commute with $\mathbb{W}^{(0)}$, which is the identity matrix. However, one has

$$\mathbb{W}^{(1)}\mathbb{W}^{(2)} = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 5 \\ 0 & 2 & 0 \end{pmatrix}, \qquad \mathbb{W}^{(2)}\mathbb{W}^{(1)} = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 2 \\ 0 & 5 & 0 \end{pmatrix}. \tag{159}$$

Therefore, $\mathbb{W}^{(1)}$ and $\mathbb{W}^{(2)}$ do not commute and, consequently, the condition (156) is not satisfied for all $F, K, E \in \bar{C}^{\infty}(\mathcal{F}_2)$, which, in turn, implies that $[\![\, ,]\!]_{\perp}$ is not a Poisson bracket on its own. We finally remark that the sum of the two contributions $[\![[\![F, K]\!]_{\perp}, E]\!]_{\perp} + \circlearrowleft$ and $[\![[\![F, K]\!]_{\perp}, E]\!]_{\|} + \circlearrowleft$ cannot vanish either, for any choice of F, K and E, because $[\![[\![F, K]\!]_{\perp}, E]\!]_{\|} + \circlearrowleft$ explicitly contains partial derivatives with respect to z of functional derivatives of F, K and E, whereas $[\![[\![F, K]\!]_{\perp}, E]\!]_{\|} + \circlearrowleft$ does not contain this type of derivatives. Therefore, there cannot be cancellation between these two contributions, for any choice of F, K and E. In conclusion, the Jacobi identity (151) is not satisfied.

We also remark that, if one had replaced the value $2\sqrt{2}$ of $\mathbb{W}^{22}_{(2)\,2}$ with the value $1/\sqrt{2}$, corresponding to $W^{22}_{(2)\,2}$, the condition (154) would have been satisfied and the three matrices $\mathbb{W}^{(0)}$, $\mathbb{W}^{(1)}$ and $\mathbb{W}^{(2)}$ would have pairwise commuted. Thus, the Jacobi identity would have been satisfied, which confirms that $\{\, , \}_g$ is a valid Poisson bracket.

The additional term, with respect to the expression (121), present in Eq. (108) for $(l, m, n) \in B_N$, is what 'corrects' the coefficient $\mathbb{W}^{mn}_{(N)\,l}$ in order to modify the form $[\![\, ,]\!]$ into a Poisson bracket. In particular, it modifies the perpendicular form $[\![\, ,]\!]_{\perp}$ turning it into $\{\, , \}_{g\perp}$, which is a Poisson bracket obtained by extension of a Lie-Poisson bracket [20].

Finally, we note that, had one naively (and not correctly) used the bilinear form (149) as a Poisson bracket, and derived a set of equations using the formula $\dot{g}_i = [\![g_i, H]\!]$, for $i = 0, 1, 2$, with H given by Eq. (139), one would have nevertheless obtained the correct equations of motion (143). Indeed, the coefficient $\mathbb{W}^{22}_{(2)\,2}$, which spoils the Jacobi identity, would have produced no finite terms in the equations of motion. Therefore, this is one of the examples that points out subtleties existing in the identification of Poisson brackets among 'almost identical' antisymmetric bilinear forms. These can indeed yield the same equations of motion from the same Hamiltonian, but they are not Poisson brackets because they do not satisfy the Jacobi identity. It also shows how truncations can spoil the Hamiltonian structure of a model.

7. Physical interpretations of the variables $G_0, G_1, \cdots, G_N$. Analogy with the problem of the quantum harmonic oscillator

The set of variables $G_0, G_1, \cdots, G_N$ introduced with Eq. (67), proved to be useful [32] in order to find the Poisson bracket for the fluid models which, in terms of these variables, takes the simple form (73). Such variables are related to the Casimir invariants of the perpendicular Poisson bracket $\{\,,\,\}_{g_\perp}$, i.e. to observables $C \in \bar{C}^\infty(\mathcal{G}_N)$ such that

$$\{C, F\}_{G_\perp} = 0, \qquad \forall F \in \bar{C}^\infty(\mathcal{G}_N). \tag{160}$$

More precisely, one has that the observables $C_i \in \bar{C}^\infty(\mathcal{G}_N)$, with $i = 0, 1, \cdots, N$, and defined by

$$C_i = \int_{\mathcal{D}} d^3x\, \mathcal{C}_i(G_i), \qquad \text{for } i = 0, 1, \cdots, N, \tag{161}$$

with $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_N$ arbitrary smooth functions, are infinite families of Casimir invariants for $\{\,,\,\}_{G_\perp}$. Casimir invariants of this form have been found in the 2D limit of a number of reduced fluid models, as for instance in those of Refs. [46, 24, 25, 26, 27]. The associated variables $G_0, G_1, \cdots, G_N$ have also been used to investigate simulations of collisionless magnetic reconnection [11, 46, 27, 10, 47] or directions of spectral cascades in 2D plasma turbulence [48]. In the 2D limit, such Casimir invariants are analogous to the Casimir invariants of the 2D Euler equation for an incompressible fluid [30]. However, in spite of their frequent occurrence in reduced fluid models for plasmas, a complete physical interpretation of the variables $G_0, G_1, \cdots, G_N$, associated with such Casimir invariants, still appears to be lacking. Here we provide an attempt to remedy this gap.

We begin by recalling, as stated at the beginning of Sec. 5.1, that the $N+1$-moment fluid model (63)-(65) can be obtained from the parent drift-kinetic system (2)-(4) by replacing g with its truncated Hermite series

$$\tilde{g}(\mathbf{x}, v, t) = \sum_{m=0}^{N} g_m(\mathbf{x}, t) \frac{H_m(v)}{\sqrt{m!}} F_{eq}(v). \tag{162}$$

Using the relation $g_m = U_{N_{mn}} G_n$, Eq. (162) can be rewritten as

$$\tilde{g}(\mathbf{x}, v, t) = \sum_{m,n=0}^{N} U_{N_{mn}} G_n(\mathbf{x}, t) \frac{H_m(v)}{\sqrt{m!}} F_{eq}(v). \tag{163}$$

Evaluating Eq. (163) at $v = \lambda_i$, for $i = 0, 1, \cdots, N$ and using the relation

$$\frac{N!}{N+1} \sum_{i=0}^{N} \frac{H_i(\lambda_j) H_i(\lambda_k)}{|H_N(\lambda_j)||H_N(\lambda_k)| i!} = \delta_{j,k}, \tag{164}$$

which, analogously to Eq. (B.3), follows from the orthogonality between U_N and U_N^T, one obtains the relation

$$\tilde{g}(\mathbf{x}, \lambda_i, t) = \sqrt{\frac{N+1}{N!}} |H_N(\lambda_i)| G_i(\mathbf{x}, t) F_{eq}(\lambda_i), \qquad \text{for } i = 0, 1, \cdots, N. \tag{165}$$

This relation can alternatively be written as

$$G_i(\mathbf{x}, t) = \alpha_{N,\lambda_i} \, \tilde{g}(\mathbf{x}, \lambda_i, t), \qquad \text{for } i = 0, 1, \cdots, N, \tag{166}$$

where

$$\alpha_{N,\lambda_i} = \sqrt{\frac{N!}{N+1}} \frac{1}{|H_N(\lambda_i)| F_{eq}(\lambda_i)}. \tag{167}$$

From the relation (166) one can then interpret each G_i as a *field providing, up to the multiplicative constant factor* α_{N,λ_i}*, the spatial and temporal evolution* *of the truncated distribution function related to electrons with parallel velocity* $v = \lambda_i$.

Using Eq. (80), the evolution equation (68) for a given G_i, can be written as

$$\frac{\partial G_i}{\partial t} + [\phi - \lambda_i A, G_i] + \lambda_i \frac{\partial G_i}{\partial z} + \sqrt{\frac{2}{\beta_e}} \sqrt{\frac{N!}{N+1}} \frac{\lambda_i}{|H_N(\lambda_i)|} \frac{\partial}{\partial z}(\lambda_i A - \phi) = 0, \quad (168)$$

where $\phi = \phi_{fl}[U_{N_{0l}} G_l]$ and $A = A_{fl}[U_{N_{1l}} G_l]$. Upon defining

$$\tilde{\mathrm{g}}_{\lambda_i}(\mathbf{x}, t) = \tilde{g}(\mathbf{x}, \lambda_i, t), \quad (169)$$

one obtains, through Eq. (165), the evolution equation for $\tilde{\mathrm{g}}_{\lambda_i}$, which reads

$$\frac{\partial \tilde{\mathrm{g}}_{\lambda_i}}{\partial t} + [\phi - \lambda_i A, \tilde{\mathrm{g}}_{\lambda_i}] + \lambda_i \frac{\partial \tilde{\mathrm{g}}_{\lambda_i}}{\partial z} + \sqrt{\frac{2}{\beta_e}} F_{eq}(\lambda_i) \lambda_i \frac{\partial}{\partial z}(\lambda_i A - \phi) = 0. \quad (170)$$

We remark that analogies between the dynamics of variables G_i in fluid models (for $N = 1$) and the dynamics of the electron drift-kinetic distribution function was observed by means of numerical simulations of magnetic reconnection in Refs. [34, 35, 36].

From the relation (165) it follows that, knowing the location of λ_i (i.e. of the zeros of $H_{N+1}(x)$), tells what are the particular values of v for which the truncated generalized perturbed distribution function is proportional to G_i. From the properties of the zeros of Hermite polynomials, some information about the location of the eigenvalues λ_i can actually be inferred.

We proceed by recalling some known properties about zeros of Hermite polynomials, from which we also draw some conclusions on the dynamics described by the reduced fluid models, and their relation with the parent drift-kinetic model.

First, for a given N, the eigenvalues λ_i are distinct [42]. Moreover, as already noticed in Sec. 5, they are symmetrically distributed around $x = 0$, and, when N is even, there exists one m such that $\lambda_m = 0$. Therefore, for fluid models with even N (i.e. evolving an odd number of moments), one has

$$\frac{\partial G_m}{\partial t} + [\phi, G_m] = 0, \quad (171)$$

i.e., there exists one, and only one field G_m, which is purely advected by the $\mathbf{E} \times \mathbf{B}$ velocity field (this was the case, for instance, of G_1 in Eq. (146)). This reflects the fact that, when N is even, the Poisson bracket (73) can be written as

$$\{F,K\}_G = \sum_{\substack{i=0\\ i\neq m}}^{N} \left(\sqrt{\frac{\beta_e}{2}} \frac{1}{U_{N_{0i}}} \int_{\mathcal{D}} d^3x\, G_i [F_{G_i}, K_{G_i}] - \lambda_i \int_{\mathcal{D}} d^3x F_{G_i} \frac{\partial K_{G_i}}{\partial z} \right)$$
$$+ \sqrt{\frac{\beta_e}{2}} \frac{1}{U_{N_{0m}}} \int_{\mathcal{D}} d^3x\, G_m [F_{G_m}, K_{G_m}]. \tag{172}$$

Casimir invariants of the bracket (172) are given by

$$\mathfrak{C}_i = \int_{\mathcal{D}} d^3x\, G_i, \qquad i = 0, \cdots, N, \qquad i \neq m, \tag{173}$$
$$\mathfrak{C}_m = \int_{\mathcal{D}} d^3x\, \mathcal{C}_m(G_m), \tag{174}$$

where $\mathcal{C}_m$ is an arbitrary smooth function. Therefore, when the number of retained moments is odd, the system admits, even in 3D, an infinite number of Casimir invariants, corresponding to the family $\mathfrak{C}_m$, in addition to the finite number of Casimir invariants $\mathfrak{C}_i$, with $i = 0, \cdots, N$ and $i \neq m$. A similar feature was encountered also in the Casimir invariants of the model described in Ref. [28].

A different situation occurs when the number of moments is even (i.e. N is odd). In this case, $\lambda_i \neq 0$ for all i and the Poisson bracket reads

$$\{F,K\}_G = \sum_{i=0}^{N} \left(\sqrt{\frac{\beta_e}{2}} \frac{1}{U_{N_{0i}}} \int_{\mathcal{D}} d^3x\, G_i [F_{G_i}, K_{G_i}] - \lambda_i \int_{\mathcal{D}} d^3x F_{G_i} \frac{\partial K_{G_i}}{\partial z} \right). \tag{175}$$

Casimir invariants thus reduce to

$$\mathfrak{C}_i = \int_{\mathcal{D}} d^3x\, G_i, \qquad i = 0, \cdots, N, \tag{176}$$

and remain in a finite number.

The explicit form of the matrix U_N provided in the present paper, makes it also now possible to easily express Casimir invariants in terms of the fluid moments, for arbitrary N. This becomes particularly relevant in the 2D case, where the evolution equations of the fluid moments can be recast in the form of advection equations for the Lagrangian invariants $G_0, G_1, \cdots, G_N$.

More in general, as it emerges from Eq. (68), the eigenvalues λ_i express the weight of the magnetic contribution, relative to the $\mathbf{E} \times \mathbf{B}$ contribution, of the generalized velocity fields

$$\mathbf{v}_i = \hat{z} \times \nabla(\phi + \lambda_i A), \qquad i = 0, 1, \cdots, N. \tag{177}$$

The incompressible velocity fields $\mathbf{v}_i$ are those that advect the fields G_i in the plane perpendicular to the guide field.

Another property of zeros of Hermite polynomials is that they interlace (see, e.g. Ref. [49]). This means that, if $\lambda_0, \lambda_1, \cdots, \lambda_N$ are the zeros of H_{N+1} and $\lambda'_0, \lambda'_1, \cdots, \lambda'_{N+1}$ are the zeros of H_{N+2}, one has

$$\lambda'_0 < \lambda_0 < \lambda'_1 < \cdots < \lambda_N < \lambda'_{N+1}. \tag{178}$$

Therefore, between two consecutive eigenvalues λ_i and λ_{i+1} of S_N, there will always be one eigenvalue λ'_{i+1} of S_{N+1}. Also, for a given interval $v_1 \leqslant v \leqslant v_2$, one can always find an eigenvalue that belongs to that interval, provided N is large enough. This suggests how the relative weight of magnetic vs. $\mathbf{E} \times \mathbf{B}$ contributions, in the generalized velocity fields $\mathbf{v}_i$, evolves, as N increases. We point out that the arguments discussed in this Section, about the eigenvalues λ_i, hold for N arbitrarily large but finite. In particular, although, as just stated, for sufficiently large N, one can find an eigenvalue λ_i arbitrarily close to a given value of v, not all the real values of v are eigenvalues of S_N, even for N arbitrarily large. Indeed, because the eigenvalues are zeros of polynomials with rational coefficients, there exists no N, for which $\lambda_i = v$, for some $i \leqslant N$, when v is a transcendental number (see also Ref. [43]). From this, we can infer a limitation in the approximation of the drift-kinetic dynamics with that of the reduced fluid models. Indeed, reduced fluid models with the adopted Hamiltonian closure, as mentioned at the beginning of Sec. 5.1, replace the actual dynamics of g with that of a truncated series. Through the relation (166), one has a direct correspondence between the variables G_i of the fluid model, and a truncated series of g, for a discrete set of $v \in \{\lambda_0, \lambda_1, \cdots, \lambda_N\}$. Thus, the dynamics of the fields G_i can be seen as an approximation of the dynamics of the actual function g, for a discrete set of values of v. However, because of the above remark on the non-transcendental character of the eigenvalues λ_i, we can conclude that there cannot be an approximation of g, by means of a certain G_i, for transcendental values of v, no matter how large N is.

We remark that the infinite hierarchy of equations (56)-(59), obtained from the drift-kinetic system without truncations, can be written as

$$\begin{aligned} &\frac{\partial g_m}{\partial t} + [\phi, g_m] - S_{mn}[A, g_n] + S_{mn}\frac{\partial g_n}{\partial z} \\ &+ \sqrt{\frac{2}{\beta_e}}\frac{\partial}{\partial z}\left(\sqrt{m!}(\delta_{m0} + \delta_{m2})A - \delta_{m1}\phi\right) = 0, \qquad m \in \mathbb{Z}_{\geqslant 0}, \end{aligned} \tag{179}$$

where $S_{mn} = \sqrt{m}\delta_{m,n+1} + \sqrt{m+1}\delta_{m,n-1}$ are the elements of the infinite matrix

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & ... & 0 & ... \\ 1 & 0 & \sqrt{2} & 0 & ... & 0 & ... \\ 0 & \sqrt{2} & 0 & \sqrt{3} & ... & 0 & ... \\ 0 & 0 & \sqrt{3} & 0 & ... & 0 & ... \\ & ... & & & ... & & ... \\ & ... & & & ... & & ... \\ 0 & 0 & 0 & ... & 0 & \sqrt{N} & ... \\ 0 & 0 & 0 & ... & \sqrt{N} & 0 & ... \\ ... & ... & ... & ... & ... & ... & ... \end{pmatrix}. \tag{180}$$

One can recognize, up to multiplicative constant factors, the matrix S as the Jacobi matrix of the position operator $\hat{x}$ for a quantum harmonic oscillator in the orthonormal basis consisting of eigenvectors of the number operator $\hat{N}$. The matrix S, in particular, can be written as $S = a^T + a$, where a^T and a are matrices of elements $a^T_{mn} = \sqrt{m}\delta_{m,n+1}$ and $a_{mn} = \sqrt{m+1}\delta_{m,n-1}$. The matrices a^T and a are associated with the ladder operators $\hat{a}^\dagger$ and $\hat{a}$, respectively. In terms of this analogy, one could interpret, from the first line Eq. (179), the temporal variation of the moment g_m as influenced by the action of such two operators. One of them (analogous to the "creation" operator $\hat{a}^\dagger$) corresponds to $\sqrt{m+1}g_{m+1}$ and the second one (analogous to the "annihilation" operator $\hat{a}$) corresponds to $\sqrt{m}g_{m-1}$. We point out that a connection between creation-annihilation operators and the dynamics of moments obtained from the Vlasov equation was investigated in Ref. [50]. In this context, we also find it interesting to mention that our approach for closing the infinite hierarchy (179), by replacing the infinite matrix S with a finite matrix S_N, is analogous to the procedure adopted in quantum mechanics for approximating the behavior of a quantum harmonic oscillator by a truncated quantum harmonic oscillator [43, 41]. Indeed, the matrices S_N correspond to those associated with the position operators of truncated harmonic oscillators. The analogy with the quantum harmonic oscillator relies on the choice of the Hermite polynomials as basis for representing the generalized perturbed distribution function g. Therefore, we believe that the analogy would fail if a different basis were chosen for representing g.

8. Conclusions

We presented new results concerning an infinite class of Hamiltonian nonlinear reduced fluid models describing the dynamics of plasma and electromagnetic fields in the presence of a strong magnetic guide field. The Hamiltonian structure of all these models is now available in an explicit form, thus completing the results of Refs. [32, 33] about the existence of such structure. Although the Hamiltonian reduced fluid models can be obtained from the parent drift-kinetic model, by truncating the Hermite series expansion of the generalized perturbed distribution function, we showed that this approach cannot be applied in order to derive the Hamiltonian structure of the fluid models from that of the parent model. Indeed, by this approach, one does not retrieve the Poisson bracket of the fluid models, but a different (although, interestingly, "very similar") bilinear operator which we showed, with a counterexample, not to satisfy, in general, the Jacobi identity. Such truncations are thus shown not to preserve the Hamiltonian structure. In order to derive the Hamiltonian structure of fluid models from that of a parent kinetic model, alternative approaches should be followed.

Examples of fluid reductions from kinetic systems, that preserve a Hamiltonian structure, are provided in Refs. [51, 52, 53, 54, 55, 56].

From a more physical perspective, we showed the existence of a relation (so far unknown, to the best of our knowledge), between the variables $G_0, G_1, \cdots, G_N$ and the truncated generalized perturbed distribution function. Also, we put in evidence some features of the dynamics of the reduced fluid models, inferred from properties of the zeros of Hermite polynomials. A limitation in the capability of a reduced fluid model to approximate the dynamics of the drift-kinetic model was also identified. Furthermore, we pointed out an analogy between the hierarchy of fluid equations and the problem of the quantum harmonic oscillator. In particular, the closure problem in the plasma physics context, shares similarities with the problem of the truncated harmonic oscillator in quantum mechanics.

In our opinion, the present paper motivates further research in various directions. On one hand, given the above mentioned failure in deriving the Poisson brackets of the reduced fluid models, by the truncated series approach, the problem of the derivation of such Poisson brackets remains open. The identification, carried out in Sec. 5, of the terms that 'correct' the coefficients $\mathbb{W}^{mn}_{(N)\,l}$, turning them into the coefficients $W^{mn}_{(N)\,l}$ of the Poisson bracket, might give some hint on how the Lie algebra underlying the Poisson bracket $\{\,,\}_g$ descends from that of the parent Poisson bracket $\{\,,\,\}_{dk}$.

A further natural direction of investigation, potentially leading to a number of applications in terms of modelling plasmas with strong anisotropies, concerns the identification of Hamiltonian closures for reduced fluid models accounting, in addition to the evolution of moments involving the coordinate v, also moments with respect to the perpendicular velocity coordinate. In Ref. [33] finite Larmor radius effects involving the perpendicular velocity (or, equivalently, the magnetic moment) coordinate, were taken into account. However, no general Hamiltonian closure was found for models evolving also moments with respect to such coordinate.

Finally, we believe it could be useful to deepen the investigation of the analogy between the present hierarchy of fluid models and the quantum harmonic oscillator. In particular, it might be interesting to see whether the techniques adopted to approximate the quantum harmonic oscillator by a truncated oscillator, as done in Ref. [41], could be transferred to the problem of approximating a drift-kinetic system by a reduced fluid model.

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AppendixA. Proof of Lemma 5.1

Proof. In order to prove the statement (a) it is convenient to introduce the function

$$\theta_{N_{lmn}}(x) = \frac{N!}{N+1} \frac{H_l(x) H_m(x) H_n(x)}{H_N^2(x)\sqrt{l!m!n!}}, \tag{A.1}$$

so that

$$W_{(N)\,l}^{mn} = \sum_{i=0}^{N} \theta_{N_{lmn}}(\lambda_i). \tag{A.2}$$

Because an Hermite polynomial $H_n(x)$ is an even (odd) function of x if n is an even (odd) number, it follows from the definition (A.1) that, if $l+m+n$ is odd, the function $\theta_{N_{lmn}}$ is odd. Moreover, the elements $\lambda_0, \lambda_1, \cdots, \lambda_N$, which are the zeros of the Hermite polynomial $H_{N+1}(x)$, are symmetrically distributed around $x = 0$ on the real axis, so that if λ_i is a zero, also $-\lambda_i$ is. In particular one has $\lambda_i = 0$, for a certain i, when H_{N+1} is an odd function, i.e. when N is even.

We first consider the case when N is odd. In this case $\lambda_i \neq 0$ for $i = 0, 1, \cdots, N$. We sort the eigenvalues λ_i in increasing order so that $\lambda_0 < \lambda_1 < \cdots < \lambda_N$. Due to the above mentioned symmetry property around $x = 0$, we have that

$$\begin{aligned} &\{\lambda_0, \lambda_1, \cdots, \lambda_{\frac{N-1}{2}}, \lambda_{\frac{N+1}{2}}, \cdots, \lambda_{N-1}, \lambda_N\} \\ &= \{-\lambda_N, -\lambda_{N-1}, \cdots, -\lambda_{\frac{N+1}{2}}, \lambda_{\frac{N+1}{2}}, \cdots, \lambda_{N-1}, \lambda_N\}. \end{aligned} \tag{A.3}$$

Thus, from Eq. (A.2), considering that $\theta_{N_{lmn}}$ is an odd function when $l+m+n$ is odd, one has

$$
\begin{aligned}
W_{(N)\,l}^{mn} &= \sum_{i=0}^{\frac{N-1}{2}} \theta_{N_{lmn}}(\lambda_i) + \sum_{i=\frac{N+1}{2}}^{N} \theta_{N_{lmn}}(\lambda_i) \\
&= \sum_{i=\frac{N+1}{2}}^{N} (\theta_{N_{lmn}}(-\lambda_i) + \theta_{N_{lmn}}(\lambda_i)) = \sum_{i=\frac{N+1}{2}}^{N} (-\theta_{N_{lmn}}(\lambda_i) + \theta_{N_{lmn}}(\lambda_i)) = 0.
\end{aligned}
\tag{A.4}
$$

The case when N is even proceeds in a similar way, with the difference that the eigenvalue $\lambda_i = 0$ must be taken into account. Therefore one has

$$
\begin{aligned}
&\{\lambda_0, \lambda_1, \cdots, \lambda_{\frac{N}{2}-1}, \lambda_{\frac{N}{2}}, \lambda_{\frac{N}{2}+1}, \cdots, \lambda_{N-1}, \lambda_N\} \\
&= \{-\lambda_N, -\lambda_{N-1}, \cdots, -\lambda_{\frac{N}{2}+1}, 0, \lambda_{\frac{N}{2}+1}, \cdots, \lambda_{N-1}, \lambda_N\}.
\end{aligned}
\tag{A.5}
$$

Analogously to Eq. (A.4), the expression for the coefficients $W_{(N)\,l}^{mn}$ is given by

$$
\begin{aligned}
W_{(N)\,l}^{mn} &= \sum_{i=0}^{\frac{N}{2}-1} \theta_{N_{lmn}}(\lambda_i) + \sum_{i=\frac{N}{2}+1}^{N} \theta_{N_{lmn}}(\lambda_i) + \theta_{N_{lmn}}(\lambda_{\frac{N}{2}}) \\
&= \sum_{i=\frac{N}{2}+1}^{N} (\theta_{N_{lmn}}(-\lambda_i) + \theta_{N_{lmn}}(\lambda_i)) + \theta_{N_{lmn}}(0) = \sum_{i=\frac{N+1}{2}}^{N} (-\theta_{N_{lmn}}(\lambda_i) + \theta_{N_{lmn}}(\lambda_i)) = 0,
\end{aligned}
\tag{A.6}
$$

where $\theta_{N_{lmn}}(0) = 0$ because $\theta_{N_{lmn}}$ is an odd function.

With regard to the statement (b), it follows straightforwardly from commutativity under multiplication, which implies $\hat{H}_{\sigma(l)}\hat{H}_{\sigma(m)}\hat{H}_{\sigma(n)} = \hat{H}_l\hat{H}_m\hat{H}_n$. Consequently, from the expression (88), we obtain

$$
\begin{aligned}
W_{(N)\,\sigma(l)}^{\sigma(m)\sigma(n)} &= \frac{N!}{N+1} \sum_{i=0}^{N} \frac{\hat{H}_{\sigma(l)}(\lambda_i)\hat{H}_{\sigma(m)}(\lambda_i)\hat{H}_{\sigma(n)}(\lambda_i)}{H_N^2(\lambda_i)} \\
&= \frac{N!}{N+1} \sum_{i=0}^{N} \frac{\hat{H}_l(\lambda_i)\hat{H}_m(\lambda_i)\hat{H}_n(\lambda_i)}{H_N^2(\lambda_i)} = W_{(N)\,l}^{mn}.
\end{aligned}
\tag{A.7}
$$

□

AppendixB. Proof of Proposition 5.2

Proof. Comparing the expressions (106), (107) and (109) with the expression (87), it immediately follows that, in order to prove the Proposition, it suffices to prove the relation (108). Also, because of Lemma 5.1 (a), we only need to consider 3-ples (l, m, n) such that $l + m + n$ is even. Moreover, by virtue of Lemma 5.1 (b), once we obtain an expression for $W_{(N)\,l}^{mn}$, we automatically obtain the expressions for all the coefficients obtained by permutations of l, m and n.

To obtain the required expression for the coefficients $W_{(N)\,l}^{mn}$ we first recall the following identity for Hermite polynomials (which can be obtained from Eq. (2.01) of Ref. [57]):

$$H_m(x)H_n(x) = \sum_{r=0}^{\min(m,n)} r! \binom{m}{r} \binom{n}{r} H_{m+n-2r}(x). \tag{B.1}$$

From Eq. (88), it follows that the coefficients $W_{(N)\,l}^{mn}$ can then be rewritten as

$$W_{(N)\,l}^{mn} = \frac{N!}{N+1} \sum_{i=0}^{N} \frac{H_l(\lambda_i)}{H_N^2(\lambda_i)} \sum_{r=0}^{\min(m,n)} \frac{m!}{(m-r)!} \frac{n!}{(n-r)!r!} \frac{H_{m+n-2r}(\lambda_i)}{\sqrt{l!m!n!}}. \tag{B.2}$$

Due to the orthogonality of the matrices U and U^T, from the expression (85), one obtains the following relation:

$$\frac{N!}{N+1} \sum_{i=0}^{N} \frac{H_j(\lambda_i)H_k(\lambda_i)}{H_N^2(\lambda_i)\sqrt{j!k!}} = \delta_{j,k}, \tag{B.3}$$

Now we consider a coefficient $W_{(N)\,l}^{mn}$ for a fixed 3-ple (l, m, n) and proceed by separating the analysis in two cases.

Case I : In a given 3-ple (l, m, n), the elements l, m, n are such that $l + m + n$ is even and there exist at least two elements such that their sum is less than or equal to $N + 1$

Let us suppose that m and n are such that $m + n \leqslant N + 1$. Due to the orthogonality relation (B.3), when the condition $m + n - 2r = l$ (with $0 \leqslant$

$r \leqslant \min(m,n))$ is satisfied, the right-hand side of Eq. (B.2) yields a finite contribution. Note also that, if $m+n = N+1$, the contribution coming from $H_{m+n}(\lambda_i) = H_{N+1}(\lambda_i)$, corresponding to $r = 0$, vanishes. This contribution, actually, is not determined by the relation (B.3), because such relation involves only Hermite polynomials up to order N. However, one has that $H_{N+1}(\lambda_i) = 0$, because, as recalled before Eq. (77), λ_i is a zero of H_{N+1} for all $i = 0, 1, \cdots, N$. Therefore, the case $m+n = N+1$ provides at most only one finite contribution to $W^{mn}_{(N)\,l}$, as does the case $m+n < N+1$. The non-zero contribution occurs for

$$r = \frac{m+n-l}{2}, \tag{B.4}$$

(recall that we are considering $l+m+n$ even, so that $m+n-l$ is also even), provided that

$$\frac{m+n-l}{2} \geqslant 0, \qquad m \geqslant \frac{m+n-l}{2}, \qquad n \geqslant \frac{m+n-l}{2}. \tag{B.5}$$

The conditions (B.5) can be reformulated as

$$m+n \geqslant l, \qquad l+m \geqslant n, \qquad n+l \geqslant m. \tag{B.6}$$

If these conditions are fulfilled, from Eq. (B.2), using Eq. (B.3), one obtains

$$\begin{aligned} W^{mn}_{(N)\,l} &= \frac{\sqrt{l!m!n!}}{\left(m - \frac{m+n-l}{2}\right)!\left(n - \frac{m+n-l}{2}\right)!\left(\frac{m+n-l}{2}\right)!} \\ &= \frac{\sqrt{l!m!n!}}{\left(\frac{l+m-n}{2}\right)!\left(\frac{n+l-m}{2}\right)!\left(\frac{m+n-l}{2}\right)!}. \end{aligned}$$

If any of the three conditions (B.6) is not satisfied, then $W^{mn}_{(N)\,l} = 0$, because there would be no r, with $0 \leqslant r \leqslant \min(m,n)$, such to provide a non-zero contribution in the right-hand side of Eq. (B.2), due to Eq. (B.3). The 3-ples (l,m,n) belonging to Case I and yielding $W^{mn}_{(N)\,l} \neq 0$ are thus those given by $(l,m,n) \in A_N \backslash B_N$.

Note that, due to the invariance of the coefficients $W^{mn}_{(N)\,l}$ under permutation, one can easily determine the coefficients in Case I also when the sum of two indices is greater than $N+1$. For instance, if one is in Case I and wants to compute $W^{nl}_{(N)\,m}$, with $n+l > N+1$, $l+m > N+1$ and $m+n \leqslant N+1$, it suffices to permute the indices and, due to $W^{nl}_{(N)\,m} = W^{mn}_{(N)\,l}$, one can follow the above procedure carried out for $W^{mn}_{(N)\,l}$.

Case II : In a given 3-ple (l, m, n), the elements l, m, n are such that $l+m+n$ is even and $l + m > N + 1, \quad m + n > N + 1, \quad n + l > N + 1$

We are thus referring to the case $(l, m, n) \in B_N$. The sum on r on the right-hand side of Eq. (B.2) involves the Hermite polynomials H_{m+n}, H_{m+n-2}, $H_{m+n-4}, \cdots, H_{m+n-2\min(m,n)}$. We denote with $r_{N_{mn}}$ the smallest integer r, with $0 < r \leqslant \min(m, n)$, such that $m + n - 2r_{N_{mn}} \leqslant N + 1$. This corresponds to the definition (104). We can then split the sum on r, in Eq. (B.2), into two parts, in the following way:

$$W^{mn}_{(N)\,l} = \frac{N!}{N+1} \sum_{i=0}^{N} \frac{H_l(\lambda_i)}{H_N^2(\lambda_i)} \left(\sum_{r=0}^{r_{N_{mn}}-1} \frac{m!}{(m-r)!} \frac{n!}{(n-r)!r!} \frac{H_{m+n-2r}(\lambda_i)}{\sqrt{l!m!n!}} \right. \tag{B.7}$$

$$\left. + \sum_{r=r_{N_{mn}}}^{\min(m,n)} \frac{m!}{(m-r)!} \frac{n!}{(n-r)!r!} \frac{H_{m+n-2r}(\lambda_i)}{\sqrt{l!m!n!}} \right). \tag{B.8}$$

The sum from $r = r_{N_{mn}}$ to $r = \min(m, n)$, in the expression (B.8), involves only Hermite polynomials of order at most equal to $N + 1$. Therefore, although $m + n > N + 1$, this expression can be treated in the same way as Case I, using the relation (B.3). On the other hand, the sum from $r = 0$ to $r = r_{N_{mn}} - 1$, in the expression (B.7), involves only Hermite polynomials of order greater than $N+1$ (and thus greater than l). For such polynomials, the orthogonality relation (B.3) does not apply. However, the terms in the expression (B.7), in general, can provide additional finite contributions to $W^{mn}_{(N)\,l}$. It follows that, from Eq. (B.7)-(B.8), the expression for $W^{mn}_{(N)\,l}$ in Case II can be written as

$$W^{mn}_{(N)\,l} = \frac{N!}{N+1} \sum_{i=0}^{N} \frac{H_l(\lambda_i)}{H_N^2(\lambda_i)} \sum_{r=0}^{r_{N_{mn}}-1} \frac{m!}{(m-r)!} \frac{n!}{(n-r)!r!} \frac{H_{m+n-2r}(\lambda_i)}{\sqrt{l!m!n!}}$$
$$+ \frac{\sqrt{l!m!n!}}{\left(\frac{l+m-n}{2}\right)! \left(\frac{n+l-m}{2}\right)! \left(\frac{m+n-l}{2}\right)!}. \tag{B.9}$$

Equation (B.9) thus yields the required expression for $W^{mn}_{(N)\,l}$ when $(l, m, n) \in B_N$.

□

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