Finding high-order analytic post-Newtonian parameters from a high-precision numerical self-force calculation

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We present a novel analytic extraction of high-order post-Newtonian (pN) parameters that govern quasicircular binary systems. Coefficients in the pN expansion of the energy of a binary system can be found from corresponding coefficients in an extreme-mass-ratio inspiral computation of the change ΔU in the redshift factor of a circular orbit at fixed angular velocity. Remarkably, by computing this essentially gauge-invariant quantity to accuracy greater than one part in 10^{225}, and by assuming that a subset of pN coefficients are rational numbers or products of π and a rational, we obtain the exact analytic coefficients. We find the previously unexpected result that the post-Newtonian expansion of ΔU (and of the change $\Delta\Omega$ in the angular velocity at fixed redshift factor) have conservative terms at half-integral pN order beginning with a 5.5 pN term. This implies the existence of a corresponding 5.5 pN term in the expansion of the energy of a binary system. Coefficients in the pN series that do not belong to the subset just described are obtained to accuracy better than 1 part in $10^{265-23n}$ at nth pN order. We work in a radiation gauge, finding the radiative part of the metric perturbation from the gauge-invariant Weyl scalar ψ_0 via a Hertz potential. We use mode-sum renormalization, and find high-order renormalization coefficients by matching a series in $L = \ell + 1/2$ to the large-L behavior of the expression for ΔU. The nonradiative parts of the perturbed metric associated with changes in mass and angular momentum are calculated in the Schwarzschild gauge.

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I. INTRODUCTION

The principal approximation methods used to compute the inspiral of compact binary systems are the post-Newtonian expansion, in which an orbital angular velocity $M\Omega$ serves as the expansion parameter; and the self-force or extreme-mass-ratio-inspiral (EMRI) approach, in which the small parameter is the mass ratio $\mathfrak{m}/M$ of the binary's two components. Previous work by Blanchet *et al.* [1,2] used an overlapping regime where both approximations are valid to check the consistency of the renormalization methods used in the two approaches and to find numerical values of pN coefficients at orders beyond the reach of current analytical work.

In the present paper, by working with much higher numerical accuracy—maintaining precision of at least one part in 10^{225} in an EMRI computation of the perturbed orbital frequency and redshift factor, and by considering orbits at much larger separation—with orbital radii extending to $10^{30}M$, we obtain two surprising results not seen in the previous study:

(1) A subset of the pN parameters in lower-order analytical work had been found to be either rationals m/n or to be sums of rationals multiplied by powers of π, the Euler constant γ and square roots of integers. Our high precision allows us to extract the exact *analytical* form of the subset of coefficients that are rationals or products of the form rational $\times\, \pi$ from our *numerical* values up to 10 pN order, corresponding to corrections smaller by $(v/c)^{22}$ than the Newtonian value.

(2) In a pN expansion, conservative terms (terms even under the interchange of outgoing and ingoing radiation) are initially encountered at integral pN orders; dissipative terms (odd under the interchange of outgoing and ingoing) first enter at 2.5 pN order. At higher order, dissipative terms can occur at either integral or half-integral order, depending on the details [3], while conservative terms enter at each integral order. We find that conservative terms of 5.5 pN order appear in the expression for the redshift at fixed angular velocity (and thus in the expressions

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for the angular velocity at fixed redshift and in the expression for the energy of an orbit with given angular velocity). These quantities are conservative, and the presence of half-integral pN terms was unexpected.

The work reported here involves a binary system that, at zeroth order in the mass ratio, is described by a test particle in a circular geodesic about a Schwarzschild black hole. At first order in $\mathfrak{m}/M$, the orbital parameters are altered by a metric perturbation $h_{\alpha\beta}$ produced by the orbiting particle: the perturbed motion can be described by saying that the particle moves on a circular geodesic of the metric $g_{\alpha\beta} + h^{\rm ren}_{\alpha\beta}$, where $h^{\rm ren}_{\alpha\beta}$ is the renormalized metric perturbation. The perturbed spacetime is helically symmetric, with a helical Killing vector k^α that is tangent to the particle's 4-velocity,

$$u^\alpha = U k^\alpha. \tag{1}$$

The constant of proportionality U is termed the redshift factor (first introduced by Detweiler [4]), and can be thought of as a contribution to the redshift, measured from the perturbed orbit of the mass m, that is independent of the internal geometry of the mass. With the perturbed spacetime chosen so that the perturbed and unperturbed helical Killing vectors coincide, the change in ΔU at fixed angular velocity Ω has the form

$$\Delta U = -U \frac{1}{2} h^{\rm ren}_{\alpha\beta} u^\alpha u^\beta =: -U H^{\rm ren}, \tag{2}$$

and it is invariant under gauge transformations generated by helically symmetric gauge vectors.

A pN expansion of ΔU, written in terms of a dimensionless $R := (M\Omega)^{-2/3}$, has the form

$$\Delta U = -\frac{1}{R} + \sum_{n=1} \alpha_n \frac{1}{R^{n+1}} + \sum_{n=4} \beta_n \frac{\log R}{R^{n+1}} + \sum_{n=7} \gamma_n \frac{\log^2 R}{R^{n+1}} + \sum_{n=10} \delta_n \frac{\log^3 R}{R^{n+1}} + \cdots, \tag{3}$$

where the post-Newtonian order n can take half-integral as well as integral values, starting at $\alpha_{5.5}$ and $\beta_{8.5}$. That is, integral values of coefficients of $\log^k R/R^{n+1}$ start at pN order $n = 3k+1$; half-integral values appear to start at $n = 3k + 5.5$, but we do not carry our numerical expansion far enough to find the first half-integral value for $k = 2$ ($\gamma_{11.5}$) or for larger k. We compute ΔU at a set of radii extending to $10^{30}M$ and match to a series of this form. As noted in the abstract, the high numerical accuracy of $\Delta U(r)$ allows us to find the coefficients α_n, β_n, and γ_n with a precision at least as high as one part in $10^{265-23n}$. At each pN order, we find that the coefficient of the highest occurring power of $\log R$ is rational when n is an integer; and it has the form rational $\times\, \pi$ when n is a half-integer. The remaining coefficients for a given value of n are not of this form.

Because the presence of $\alpha_{5.5}$ and higher-order half-integral coefficients was not expected, we performed an elaborate set of checks. Our calculations were carried out in a radiation gauge, but we repeated the entire numerical calculation of ΔU in a Regge-Wheeler gauge, obtaining numerical agreement to 368 places of accuracy; that is, the retarded values of h_{uu} for each ℓ mode in the radiation gauge and the Regge-Wheeler-Zerilli gauge agree to more than 368 digits. This serves as a demanding test of both the numerical code and of the analytical computation on which it is based. Because the numerical calculation is performed in Mathematica, the comparison is also a check of Mathematica's claimed numerical precision. Adrian Ottewill and Marc Casals kindly used their codes to perform an independent radiation-gauge computation to compare with ours at double-precision accuracy for small R. Specifically, for the $s = \ell = m = 2$ term, we compared our values of the invariant, $A_{lm} R_H R_\infty$ [see Eq. (7) below], at $r/M = 10^3, 10^6$. Finally, we analytically computed $\alpha_{5.5}$ (see Sec. III A).

In Sec. II we briefly review the calculation of the renormalized ΔU in a modified radiation gauge. In Sec. III we present the results of matching a sequence of values $\Delta U(r)$ to a series of the form (3).

We work in gravitational units ($G = c = 1$) and use signature $+---$ to conform to Newman-Penrose conventions.

II. REVIEW OF ΔU COMPUTATION

We consider a particle of mass $\mathfrak{m}$ orbiting a Schwarzschild black hole of mass M. At zeroth order in $\mathfrak{m}/M$, the trajectory is a circular orbit of radius r_0. In Schwarzschild coordinates, its angular velocity is $\Omega = \sqrt{M/r_0^3}$, and its 4-velocity is given by

$$u^\alpha = U(t^\alpha + \Omega \varphi^\alpha), \quad \text{with} \quad U = u^t = \frac{1}{\sqrt{1 - 3M/r_0}}. \tag{4}$$

We compute the change ΔU at first order in $\mathfrak{m}/M$ in a modified radiation gauge, as detailed in [5]. We briefly review the formalism here, noting first that Eq. (3) for ΔU involves a single component $H^{\rm ren}$ of the renormalized metric perturbation.

For multipoles with $\ell \geq 2$, the metric perturbation can be found in a radiation gauge from the the spin-2 retarded Weyl scalar, ψ_0, which has the form [5–7],

$$\psi_0(x) = \psi_0^{(0)} + \psi_0^{(1)} + \psi_0^{(2)}, \tag{5}$$

with

$$\psi_0^{(0)} = 4\pi\mathfrak{m}u^t\frac{\Delta_0^2}{r_0^2}\sum_{\ell m}A_{\ell m}[(\ell-1)\ell(\ell+1)(\ell+2)]^{1/2}R_{\rm H}(r_<)R_\infty(r_>)_2Y_{\ell m}(\theta,\varphi)\bar{Y}_{\ell m}\left(\frac{\pi}{2},\Omega t\right), \tag{6a}$$

$$\psi_0^{(1)} = 8\pi i\mathfrak{m}\Omega u^t\Delta_0\sum_{\ell m}A_{\ell m}[(\ell-1)(\ell+2)]^{1/2}{}_2Y_{\ell m}(\theta,\varphi)_1\bar{Y}_{\ell m}\left(\frac{\pi}{2},\Omega t\right) \times\left\{[im\Omega r_0^2+2r_0]R_{\rm H}(r_<)R_\infty(r_>)+\Delta_0[R'_{\rm H}(r_0)R_\infty(r)\theta(r-r_0)+R_{\rm H}(r)R'_\infty(r_0)\theta(r_0-r)]\right\}, \tag{6b}$$

$$\begin{aligned}\psi_0^{(2)} = {}&-4\pi\mathfrak{m}\Omega^2u^t\sum_{\ell m}A_{\ell m2}Y_{\ell m}(\theta,\phi)_2\bar{Y}_{\ell m}\left(\frac{\pi}{2},\Omega t\right)\\ &\times\{[30r_0^4-80Mr_0^3+48M^2r_0^2-m^2\Omega^2r_0^6-2\Delta_0^2-24\Delta_0r_0(r_0-M)+6im\Omega r_0^4(r_0-M)]R_{\rm H}(r_<)R_\infty(r_>)\\ &+2(6r_0^5-20Mr_0^4+16M^2r_0^3-3r_0\Delta_0^2+im\Omega\Delta_0r_0^4)[R'_{\rm H}(r_0)R_\infty(r)\theta(r-r_0)+R'_\infty(r_0)R_{\rm H}(r)\theta(r_0-r)]re\\ &+r_0^2\Delta_0^2[R''_{\rm H}(r_0)R_\infty(r)\theta(r-r_0)+R''_\infty(r_0)R_{\rm H}(r)\theta(r_0-r)+{\rm W}[R_{\rm H}(r),R_\infty(r)]\delta(r-r_0)]\},\end{aligned} \tag{6c}$$

where $\Delta = r^2 - 2Mr$; the functions R_H and R_∞ (indices ℓ, m are suppressed) are the solutions to the homogenous radial Teukolsky equation that are ingoing and outgoing at the future event horizon and null infinity, respectively, and a prime denotes their derivative with respect to r; ${\rm W}[R_{\rm H}(r), R_\infty(r)] = R_{\rm H}R'_\infty - R_\infty R'_{\rm H}$; and the quantities $A_{\ell m}$, given by

$$A_{\ell m} := \frac{1}{\Delta^3{\rm W}[R_{\rm H}(r), R_\infty(r)]}, \tag{7}$$

are constants, independent of r. The functions R_H and R_∞ are calculated to more than 350 digits of accuracy using expansions in terms of hypergeometric functions given in [8], namely,

$$R_H = e^{i\epsilon x}(-x)^{-2-i\epsilon}\sum_{n=-\infty}^{\infty}a_nF(n+\nu+1-i\epsilon, -n-\nu-i\epsilon, -1-2i\epsilon; x), \tag{8}$$

$$R_\infty = e^{iz}z^{\nu-2}\sum_{n=-\infty}^{\infty}(-2z)^nb_nU(n+\nu+3-i\epsilon, 2n+2\nu+2; -2iz), \tag{9}$$

where $x = 1-\frac{r}{2M}$, $\epsilon = 2Mm\Omega$, and $z = -\epsilon x$. We refer the reader to [8,9] for the derivation of ν (the renormalized angular momentum), and the coefficients a_n and b_n. Here F and U are the hypergeometric and the (Tricomi's) confluent hypergeometric functions.

The computation of the spin-weighted spherical harmonics ${}_sY_{\ell,m}(\theta,\varphi)$ is done analytically using [7].

Once ψ_0 is computed, the components of the metric perturbation are found from the Hertz potential, Ψ, whose angular harmonics are related to those of ψ_0 by an algebraic equation,

$$\Psi_{\ell m} = 8\frac{(-1)^m(\ell+2)(\ell+1)\ell(\ell-1)\bar{\psi}_{\ell,-m}+12imM\Omega\psi_{\ell m}}{[(\ell+2)(\ell+1)\ell(\ell-1)]^2+144m^2M^2\Omega^2} \tag{10}$$

where $\Psi = \sum_{\ell,m}\Psi_{\ell m}(r)_2Y_{\ell m}(\theta,\phi)e^{-im\Omega t}$ and $\psi_0 = \sum_{\ell,m}\psi_{\ell m}(r)_2Y_{\ell m}(\theta,\phi)e^{-im\Omega t}$. The components of the metric along the Kinnersley tetrad are

$$h_{\mathbf{11}} = \frac{r^2}{2}(\bar{\eth}^2\Psi + \eth^2\bar{\Psi}), \tag{11}$$

$$h_{\mathbf{33}} = r^4\left[\frac{\partial_t^2-2f\partial_t\partial_r+f^2\partial_r^2}{4} - \frac{3(r-M)}{2r^2}\partial_t + \frac{f(3r-2M)}{2r^2}\partial_r + \frac{r^2-2M^2}{r^4}\right]\Psi, \tag{12}$$

$$h_{13} = -\frac{r^3}{2\sqrt{2}}\left(\partial_t - f\partial_r - \frac{2}{r}\right)\bar{\eth}\Psi, \tag{13}$$

where $f = \Delta/r^2$ and the operators ð and $\bar{\eth}$, acting on a spin-s quantity, η, are given by

$$\eth\eta = -(\partial_\theta + i\csc\theta\partial_\phi - s\cot\theta)\eta,$$
$$\bar{\eth}\eta = -(\partial_\theta - i\csc\theta\partial_\phi + s\cot\theta)\eta. \tag{14}$$

The metric recovered from $\psi_0^{\rm ren}$ above only specifies the radiative part of the perturbations ($\ell \geq 2$) and the full metric reconstruction requires one to take into account the changes in mass and angular momentum of the Schwarzschild metric which are associated with $\ell = 0$ and $\ell = 1$ harmonics, respectively. The contribution to the full H from the change in mass ($H_{\delta M}$) and angular momentum ($H_{\delta J}$) of the Schwarzschild metric are given by [see Eqs. (137)–(138) of [5]]

$$H_{\delta M} = \frac{\mathfrak{m}(r_0 - 2M)}{r_0^{1/2}(r_0 - 3M)^{3/2}}, \tag{15}$$

$$H_{\delta J} = \frac{-2M\mathfrak{m}}{r_0^{1/2}(r_0 - 3M)^{3/2}}. \tag{16}$$

The renormalization of H is described in detail in [5–7]. The related quantity $\Delta\Omega$ that gives the $O(\mathfrak{m})$ change in the angular velocity of a trajectory at fixed redshift factor is

$$\Delta\Omega = -\frac{1}{u_\phi u^t}H^{\rm ren} = \frac{\Delta U}{u_\phi u^{t^2}}. \tag{17}$$

III. RESULTS

In this section we present the pN coefficients of ΔU. Prior to this work, the following analytical coefficients were known [1,10]:

$$\Delta U = \frac{-1}{R} + \frac{-2}{R^2} + \frac{-5}{R^3} + \frac{-3872 + 123\pi^2}{96R^4} + \frac{-592384 - 196608\gamma + 10155\pi^2 - 393216\log(2)}{7680R^5} + \frac{64\log(R)}{5R^5} + \frac{-956\log(R)}{105R^6}. \tag{18}$$

We calculate ΔU for a set of R values from $1\times, 3\times, 5\times, 8\times 10^{20}$ to 10^{29} in logarithmic intervals of 10 with an accuracy of one part in 10^{227} for $R = 10^{20}$, 10^{242} for $R = 10^{25}$ and 10^{252} for $R = 10^{30}$. We then match this data to a pN series to extract the unknown coefficients. In doing so, we find nonzero half-integer ($n.5$) pN coefficients that come from the tail-of-tail terms in pN computations [11]. To confirm its presence we analytically calculated the 5.5 pN term, the coefficient of $1/R^{6.5}$, and found that it agreed with the numerically extracted coefficient to 113 significant digits. (The analytic calculation is described briefly below.) The high accuracy of the numerically extracted coefficients, however, allows us to extract their exact analytical expressions, *without an analytic calculation*. For example, the numerically extracted value of the 6-pN log term is

$$-90.398589065255731922398589065255731922398589065255731922$$
$$398589065255731922398589065255731922398589048525187995\ldots \tag{19}$$

More than five repetition cycles of the string 398589065255731922 tells us that it is the rational number $-51256/567$. In a similar fashion we extract analytical values of other coefficients making the pN series of analytically known coefficients the following:

$$\begin{aligned}\Delta U_{\text{analytically known}} = {}& \frac{-1}{R} + \frac{-2}{R^2} + \frac{-5}{R^3} + \frac{-3872 + 123\pi^2}{96R^4} + \frac{-592384 - 196608\gamma + 10155\pi^2 - 393216\log(2)}{7680R^5} \\ & + \frac{64\log(R)}{5R^5} + \frac{-956\log(R)}{105R^6} + \frac{-13696\pi}{525R^{6.5}} + \frac{-51256\log(R)}{567R^7} + \frac{81077\pi}{3675R^{7.5}} + \frac{27392\log^2(R)}{525R^8} \\ & + \frac{82561159\pi}{467775R^{8.5}} + \frac{-27016\log^2(R)}{2205R^9} + \frac{-11723776\pi\log(R)}{55125R^{9.5}} + \frac{-4027582708\log^2(R)}{9823275R^{10}} \\ & + \frac{99186502\pi\log(R)}{1157625R^{10.5}} + \frac{23447552\log^3(R)}{165375R^{11}}.\end{aligned} \tag{20}$$

A rational number with fewer than ten digits in its numerator and in its denominator is determined by the first eleven digits in its decimal expansion; thus if one assumes that the rationals occurring in the coefficients of (20) have this character, they are uniquely determined by the numerical accuracy. Without the assumption, the probability that the first n digits in a decimal representation of a randomly chosen number will match a rational with n_n and n_d digits in numerator and denominator is less than $10^{n_n+n_d-n}$, when $n > n_n + n_d$.

TABLE I. Numerical values of the coefficients in the expansion (2) of ΔU for which analytic expressions could not be inferred.

Coefficient	Numerical value
α_5	−243.176814464674307587293588966938002347372728172327865395288683088279481305578784400882095188756492605696582771045263777303802870480[a]
α_6	−1305.0013810787096557410900682717136851595808847394760333078920251334987769059271121798252271389605769024318 54[a]
α_7	−6343.874453199030652727051206605306139044604629518769203158132865789206393048289 2366
α_8	−11903.47294720130441597586856241408269022857453416201732 22629
$\alpha_{8.5}$	−8301.37370829085581136384718573193317705504946743
α_9	−32239.6275950925564123677060345920962
$\alpha_{9.5}$	−10864.6255867062440752457 67
α_{10}	−221316.52514302
$\alpha_{10.5}$	6.035×10^4
β_7	536.405212471024286871789539475038911270206269552321207927883360240368736326766131833[a]
β_8	1490.555085695890743801197409898839516699272431113593795047 47[a]
β_9	−3176.929181153969206392338832692666088
β_{10}	−7358.271055677
$\beta_{10.5}$	5013.2
γ_{10}	2105.92718670257

[a]See *Note added* at the end of the paper.

After using the above analytical coefficients, a numerical fit for the other numerical coefficients in (3) gives the values listed in Table I.

A. Exact, analytic 5.5 pN value

As mentioned above, as a check on the work, we analytically compute the 5.5 pN term. To do so, we use the fact that the renormalization parameters that characterize the singular part of $H^{\rm ret}$ have *no* n.5 pN terms: the pN expansion of $H^{\rm sing}$ does not include half-integer powers of $1/R$. Studying the pN expansion of the first few multipoles of $H^{\rm ret}$, we find that the 5.5 pN term comes only from the $\ell = 2, m = \pm 2$ multipole of $H^{\rm ret}$. That is, the numerical coefficient of the 5.5 pN term we obtain by matching $H^{\rm ren}$ coincides exactly with that obtained by matching the sum of the $\ell = 2, m = \pm 2$ multipoles of $H^{\rm ret}$ to a pN series. The analytic calculation was thus restricted to the $\ell = 2$, $m = \pm 2$ multipoles of $R_H^{(p)} R_\infty^{(q)} / (R_H R'_\infty - R'_H R_\infty)$ (where p and q, the number of radial derivatives, each run from 0 to 2). We use the hypergeometric series Eqs. (8) and (9) to express each of these functions as Taylor series in powers of $1/R$. From these series, we obtain in turn the pN expansions of the $l = 2$, $m = \pm 2$ contributions to ψ_0, Ψ and their first two radial derivatives and, finally, the pN series of $H^{\rm ret}_{2,\pm 2}$.

IV. NUMERICAL EXTRACTION AND ERROR ANALYSIS

We describe in this section the way we numerically extract the pN coefficients and check the accuracy with which they are determined. We compute $\Delta U(R)$ for $R = 1\times, 3\times, 5\times, 8 \times 10^{20}$ to 10^{29} in logarithmic intervals of 10. From this data, after subtracting the known terms of Eq. (18), we match it to

$$\sum_{n=5} \alpha_n \frac{1}{R^{n+1}} + \sum_{n=6} \beta_n \frac{\log R}{R^{n+1}} + \sum_{n=7} \gamma_n \frac{\log^2 R}{R^{n+1}} + \sum_{n=10} \delta_n \frac{\log^3 R}{R^{n+1}} + \cdots. \tag{21}$$

The accuracy with which the coefficients are extracted depends on the number of terms in the series. The fit is done in Mathematica, and, for each extracted coefficient, the number of terms kept in each of the series in Eq. (21) is chosen to maximize the accuracy of that coefficient. To illustrate the procedure, let k be highest power of R^{-1} kept in a fitting series, and denote by $\alpha_{5.5}(k)$ the value of $\alpha_{5.5}$ obtained by truncating the series after R^{-k}. To extract the value of $\alpha_{5.5}$, we then look at the fractional difference $|\alpha_{5.5}(k \pm 1)/\alpha_{5.5}(k) - 1|$, find the cutoff $k = k_0$ at which the fractional difference is minimum, and use the value $\alpha_{5.5}(k_0)$. For further details we refer the reader to Sec. V of [7] where a similar fitting is done. The fitting procedure is done twice here: we first extract the new analytical pN coefficients [the terms in Eq. (20) minus Eq. (18)]; we then subtract the analytical coefficients from the data and do another set of fits to extract the coefficients in Table I.

V. DISCUSSION

In [12] it was established that a relation exists between, on the one hand, coefficients in the pN expansion of the redshift variable and, on the other hand, coefficients in the expansion of the pN binding energy and angular momentum for the binary system; for explicit results, see, for example, Eqs. (2.50a)–(2.50d) and (4.25a)–(4.25d) in [12]. Subsequently, using essentially Eqs. (2.40), (4.19) and (4.23) in [12], Le Tiec *et al.* (see [13]) transformed these relations to obtain, for arbitrary pN order, elegant expressions for the energy and angular momentum directly in terms of the self-force redshift variable and its first derivative; see, in particular, Eqs. (4a)–(4b) in [13].

Self-force extensions of the pN binding energy and angular momentum have been long sought after, since they were known to have the potential to contribute to the effective-one-body (EOB) formulation (see [14,15]) of the binary inspiral problem—mimicking, as far as possible, the reduced mass form of the Newtonian problem, but in a fully four-dimensional, space-time setting. Thus, in a follow-up paper to [13], Barausse *et al.* [16] found a very compact result, expressing the relevant EOB function directly in terms of the self-force variable alone; see their Eq. (2.14) for this important relation, subsequently also reported in [17].

There is a very clear synergy between self-force results, and their applications in pN and EOB work, and knowledge of our new results will have an immediate impact though the application of the relations discussed in the previous two paragraphs. Since the completion of our calculation, a corresponding computation has been performed to directly evaluate the 5.5 pN coefficient through conventional pN analysis, in which it is known to arise from a tails-of-tails contribution. The ensuing result, as reported in a companion paper [11], is in exact agreement with the 5.5 pN term in our Eq. (20).

The reader should be aware that the works cited in this section express Ω as $x^{3/2}/(M+\mathfrak{m})$ rather than our $R^{3/2}/M$, and use $z(x) = 1/U(R)$ as the redshift variable. The notation used throughout the rest of this paper was first introduced by Detweiler [4].

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Note added.—We recently received email notification from Nathan Kieran Johnson-McDaniel [18] that α_5 could be represented as

$$\frac{205680256 + 7342080\gamma - 31680075\pi^2 + 28968960\log(2) - 13996800\log(3)}{403200}.$$

An equivalent result and an exact expression for α_6 have subsequently appeared in [19]. It has since been possible to show that our numerical results for β_7 and β_8 can be represented by

$$\beta_7 = \frac{5163722519}{5457375} - \frac{109568}{525}\gamma - \frac{219136}{525}\log(2)$$

and

$$\beta_8 = \frac{769841899153}{496621125} + \frac{108064}{2205}\gamma + \frac{1787104}{3675}\log(2) - \frac{18954}{49}\log(3).$$

An explanation of these results and the methods used to obtain them will be discussed in a forthcoming paper [20].

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