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The post-Newtonian (PN) approximation is still the most widely used approach to obtaining explicit solutions in general relativity, especially for the relativistic two-body problem with arbitrary mass ratio. Within many of its applications, it is often required to use a regularization procedure. Though frequently misunderstood, the regularization is essential for waveform generation without reference to the internal structure of orbiting bodies. In recent years, direct comparison with the self-force approach, constructed specifically for highly relativistic particles in the extreme mass ratio limit, has enabled preliminary confirmation of the foundations of both computational methods, including their very independent regularization procedures, with high numerical precision. In this paper, we build upon earlier work to carry this comparison still further, by examining next-to-next-to-leading order contributions beyond the half integral 5.5PN conservative effect, which arise from terms to cubic and higher orders in the metric and its multipole moments, thus extending scrutiny of the post-Newtonian methods to one of the highest orders yet achieved. We do this by explicitly constructing tail-of-tail terms at 6.5 and 7.5PN order, computing the redshift factor for compact binaries in the small mass ratio limit, and comparing directly with numerically and analytically computed terms in the self-force approach, obtained using solutions for metric perturbations in the Schwarzschild space-time, and a combination of exact series representations possibly with more typical PN expansions. While self-force results may be relativistic but with restricted mass ratio, our methods, valid primarily in the weak-field slowly moving regime, are nevertheless in principle applicable for arbitrary mass ratios.

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I. INTRODUCTION

Over the last five years, comparison between post-Newtonian (PN) and gravitational self-force calculations has made rapid progress, in large part due to both high precision numerical computations from a self-force perspective [1–8] (either by directly linearizing the Einstein field equations or by using the Teukolsky equation [9–11] or the Regge-Wheeler and Zerilli equations [12,13]), and extensive analytical computations within the post-Newtonian approximation [3,4,14]. Much more recently, the possibility for this comparison has been dramatically extended. From the self-force side [15–18], this is due to the new application of (already more than 15 years old) techniques [19–21] with which to represent metric perturbation solutions for black hole space-times. On the post-Newtonian side, this has required the computation of previously unevaluated higher order terms including tail-of-tail effects [22] and, in particular, half-integral $\frac{n}{2}$PN terms that are nevertheless conservative. In this paper, we extend that most recent work. As will be seen, although the computations are indeed very extensive, the results are quite simple to state and, along with further motivation, they are listed below, before we describe in detail the processes we have used in their derivation.

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A. Motivation

The self-force problem concerns itself with computations for binary orbiting systems composed of compact bodies in which the mass ratio is extreme, such that a full numerical relativity approach is unfeasible, due to the vastly different length scales associated with the very different masses and physical sizes of the compact bodies. Foundations for the gravitational self-force (GSF) computations of compact binaries have developed over the last two decades [23–27] (see Refs. [28–30] for reviews), following very early work by De Witt and Brehme more than half a century ago [31]. For the conservative part of the dynamics, this has led to the recent possibility of high-order comparisons between self-force computations [1,3,4], on the one hand, and traditional post-Newtonian calculations (reviewed in Ref. [32]) on the other hand, with ever increasing precision.

For compact binaries moving on exactly circular orbits, Detweiler [1] introduced a gauge invariant redshift factor, computed it numerically, and showed agreement with existing PN analytical calculations [33] up to 2PN order. Then a systematic program of comparison was initiated in Refs. [3,4] which showed that GSF and PN methods agree for the 3PN term and specific logarithmic tail-induced contributions arising at 4PN and 5PN orders, and predicted numerically the values of high-order PN coefficients, notably the full 4PN coefficient. The analytical 4PN coefficient was then obtained [16] using a combination of analytical self force (SF) computation and a partial derivation of the 4PN equations of motion in the Arnowitt-Deser-Misner Hamiltonian formalism [34,35], with very good agreement with the numerical value computed in Ref. [4].

Since that work, the accuracy of the numerical computation of the GSF has improved drastically [15]. The PN coefficients of the redshift factor were obtained numerically to 10.5PN order and for a subset of coefficients, also analytically, specifically those that are either rational, made of the product of π with a rational, or a simple sum of commonly occurring transcendentals [15]. An alternative self-force approach [17,18] (based on the post-Minkowskian expansion of the Regge-Wheeler-Zerilli (RWZ) equation following Refs. [19–21]) has also obtained high-order PN coefficients analytically, up to 8.5PN order.

A feature of the post-Newtonian expansion at high order is the appearance of *half-integral* PN coefficients (of type $\frac{n}{2}$PN where n is an odd integer) in the conservative dynamics of binary point particles, moving on exactly circular orbits. Using standard post-Newtonian methods it was proved [22][1] that the dominant half-integral PN term occurs at the 5.5PN order (confirming the finding of Ref. [15]) and originates from the nonlinear tail-of-tail integrals [36]. Here we continue Paper I and compute, still using the traditional PN method (in principle applicable for any mass ratio), high-order half-integral PN terms at orders 6.5PN and 7.5PN in the redshift factor, thus corresponding to the next-to-next-to-leading half-integral contributions.

B. Results

We have computed the redshift factor introduced in Ref. [1], for a particle moving on an exact circular orbit around a Schwarzschild black hole. The ensuing space-time is helically symmetric, with a helical Killing vector K^α such that its value K_1^α at the location of the particle is proportional to the normalized four-velocity u_1^α of the particle,

$$u_1^\alpha = u_1^T K_1^\alpha. \tag{1.1}$$

The redshift factor, denoted u_1^T, is thus defined geometrically as the conserved quantity associated with the helical Killing symmetry appropriate to conservative space-times with circular orbits. However, adopting a coordinate system in which the helical Killing vector reads $K^\alpha\partial_\alpha = \partial_t + \Omega\partial_\varphi$, where Ω is the orbital frequency of the circular motion, the redshift factor reduces to the t component $\mathrm{d}t/\mathrm{d}\tau_1$ of the particle's four-velocity (where $\mathrm{d}\tau_1$ is the particle's proper time), and is thereby obtained as

$$u_1^T = \left[-g_{\alpha\beta}(y_1)\frac{v_1^\alpha v_1^\beta}{c^2}\right]^{-1/2}, \tag{1.2}$$

where $g_{\alpha\beta}(y_1)$ is the regularized metric evaluated at the particle's location $y_1^\alpha = (ct, y_1^i)$, which we shall compute in detail in the present paper for insertion into the redshift factor (1.2), and where $v_1^\alpha = \mathrm{d}y_1^\alpha/\mathrm{d}t = (c, v_1^i)$ is the coordinate velocity.

In a first stage, our calculation is valid for a general extended matter source, in the vacuum region outside the source. Then, in a second stage, we use a matching argument to continue that solution inside the source, which is then specialized to a binary point particle system. Finally the metric is evaluated at the location of one of the particles, with the help of a self-field regularization, in principle dimensional regularization. Using the relative frame of the center of mass and reducing the expressions to circular orbits, mindful of the modification of the relation between the orbital separation and the orbital frequency, we finally obtain the redshift factor in the limit of a small mass ratio $q = m_1/m_2$ (where m_1 is the small particle and m_2 is the black hole). In the test-mass limit the redshift factor is given by the Schwarzschild value,

$$u^T_{\mathrm{Schw}} = \frac{1}{\sqrt{1-3y}}, \tag{1.3}$$

where $y = (Gm_2\Omega/c^3)^{2/3}$ is the frequency-related parameter associated with the motion of the test-mass particle around the black hole. The self-force part to the redshift factor u^T_{SF} is then defined as $u_1^T = u^T_{\mathrm{Schw}} + q u^T_{\mathrm{SF}} + \mathcal{O}(q^2)$. We finally find that the half-integral conservative contributions therein up to 2PN relative order are

$$u^T_{\mathrm{SF}} = -y - 2y^2 - 5y^3 + \cdots - \frac{13696}{525}\pi y^{13/2} + \frac{81077}{3675}\pi y^{15/2} + \frac{82561159}{467775}\pi y^{17/2} + \cdots, \tag{1.4}$$

where we have written only the relative 2PN terms relevant to our next-to-next-to-leading order calculation, i.e. the Newtonian, 1PN and 2PN terms for the dominant effects, and the 5.5PN, 6.5PN and 7.5PN terms for the half-integral conservative corrections, with all the other terms, not computed in the present work, indicated by ellipsis.[2]

[1]Hereafter we refer to this paper as Paper I.

[2]The sign of the Newtonian term in Eq. (5.18) of Paper I should be changed and read $-y$.

The result (1.4) is in full agreement with results derived by gravitational self-force methods, either numerical, semi-analytical or purely analytical [15,17,18].

Let us emphasize again that the result (1.4) has been achieved from the traditional post-Newtonian approach. Contrary to various analytical and numerical self-force calculations [15,17,18] the PN approach is completely general, i.e. it is not tuned to a particular type of source as it is applicable to any extended post-Newtonian source with spatial compact support. It is remarkable that this general method can nevertheless be specialized to such a degree that it is able to control terms up to the very high order 7.5PN.

With the post-Newtonian coefficients in the redshift factor (1.4), one can straightforwardly obtain, by making use of the first law of black hole binary mechanics [14], the corresponding coefficients in the PN binding energy and angular momentum of the system [37] and the most important effective-one-body potential [8,38].

In the remainder of this paper, we first discuss vacuum solutions in the exterior zone (Sec. II). Then we investigate tail-of-tail terms in the near zone (Sec. III), listing the terms which need to be evaluated, and introduce a gauge transformation to shorten the subsequent calculation. In Sec. IV, we set up the PN iteration of tails of tails, then compute the quadratic and cubic contributions in turn. We end with a brief discussion of our results (Sec. V), with Appendix A providing an alternative derivation of some key results, and with Appendix B providing the source terms required for our tail-of-tail calculations.

II. SOLVING THE EINSTEIN EQUATIONS IN THE EXTERIOR ZONE

In the present paper we shall continue and extend the method of Paper I. Namely we compute a series of nonlinear tail effects in the exterior vacuum region around a general isolated source. We show that a crucial piece in the expansion of these nonlinear tails can be extended using a matching argument from the near zone of the source to the inner region of the source, while the other pieces will not contribute to the half-integral post-Newtonian orders in which we are interested. This crucial piece is then specialized to the case of point particle binaries and evaluated at the very location of one of the particles. Finally the corresponding metric is inserted into the redshift factor of that particle and the small mass ratio limit is computed in order to obtain the self-force prediction which is meaningfully compared to direct analytical or numerical self-force calculations.

The vacuum exterior field of a general source is computed using the multipolar–post-Minkowskian (algorithm [36,39,40], i.e. decomposed into multipolar spherical harmonics and iterated in a nonlinear or post-Minkowskian way. Using harmonic coordinates, the equation that we have to solve at each post-Minkowskian order is a (flat) d'Alembertian equation for the components of the gothic metric deviation, whose right-hand side is known from previous iterations. Furthermore, if we project out that equation on a basis of multipolar spherical harmonics with multipole index ℓ, we end up solving a generic equation of the type

$$\Box u_L(\mathbf{x},t) = \hat{n}_L S(r, t-r/c). \tag{2.1}$$

Here $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the flat space-time d'Alembertian operator, $r = |\mathbf{x}|$ is the coordinate distance from the field point to the origin located inside the matter source, and $\hat{n}_L$ is a symmetric-trace-free (STF) product of ℓ unit vectors $n_i = x_i/r$, which is equivalent to the usual basis of spherical harmonics.[3] The solution of Eq. (2.1) for a source term S which tends to zero sufficiently rapidly when $r \to 0$ (see the precise conditions in Ref. [39]) reads

$$u_L(\mathbf{x},t) = c\int_{-\infty}^{t-r/c} \mathrm{d}s\,\hat{\partial}_L\left\{\frac{1}{r}\left[R\left(\frac{t-s-r/c}{2}, s\right) - R\left(\frac{t-s+r/c}{2}, s\right)\right]\right\}, \tag{2.2}$$

where $R(\rho,s)$ denotes some intermediate function defined in terms of the source by

$$R(\rho,s) = \rho^\ell \int_0^\rho \mathrm{d}\lambda \frac{(\rho-\lambda)^\ell}{\ell!}\left(\frac{2}{\lambda}\right)^{\ell-1} S(\lambda,s). \tag{2.3}$$

For the present work, since we shall perform a matching of this solution to the inner field of a post-Newtonian source, we shall need the expansion of the solution (2.2) in the near zone, i.e. when $r \to 0$ formally. Denoting with an overbar the formal expansion when $r \to 0$ we can write the following crucial formula [41]:

$$\overline{u_L(\mathbf{x},t)} = \hat{\partial}_L\left\{\frac{\overline{G(t-r/c) - G(t+r/c)}}{r}\right\} + \Box^{-1}_{\text{inst}}[\hat{n}_L \overline{S(r,t-r/c)}]. \tag{2.4}$$

The second term in that formula represents a particular solution of the equation (2.1), in the form of an expansion when $r \to 0$, and given by the so-called operator of the instantaneous potentials defined by

[3] For STF tensors we use the same notation as in Paper I: $L = i_1 \cdots i_\ell$ denotes a multi-index composed of ℓ spatial indices ranging from 1 to 3; similarly $L-1 = i_1 \cdots i_{\ell-1}$; $\partial_L = \partial_{i_1} \cdots \partial_{i_\ell}$ is the product of ℓ partial derivatives $\partial_i \equiv \partial/\partial x^i$; $x_L = x_{i_1} \cdots x_{i_\ell}$ is the product of ℓ spatial positions x_i; $n_L = n_{i_1} \cdots n_{i_\ell}$ is the product of ℓ unit vectors $n_i = x_i/r$; the STF projection is indicated with a hat, i.e. $\hat{x}_L \equiv \mathrm{STF}[x_L]$, $\hat{n}_L \equiv \mathrm{STF}[n_L]$, $\hat{\partial}_L \equiv \mathrm{STF}[\partial_L]$ (for instance $\hat{\partial}_{ij} = \partial_{ij} - \frac{1}{3}\delta_{ij}\Delta$), or sometimes with angular brackets surrounding the indices, e.g. $x_{\langle i_\ell}\partial_{L-1\rangle} \equiv \mathrm{STF}[x_{i_\ell}\partial_{L-1}]$; in the case of summed-up multi-indices L, we do not write the ℓ summations from 1 to 3 over the dummy indices.

$$\Box^{-1}_{\mathrm{inst}}[\hat{n}_L\overline{S(r,t-r/c)}] = \sum_{i=0}^{+\infty}\left(\frac{\partial}{c\partial t}\right)^{2i}\Delta^{-1-i}[\hat{n}_L\overline{S(r,t-r/c)}]. \tag{2.5}$$

Note that such an operator acts directly (term by term) on the formal expansion of the source when $r\to 0$, given by the usual Tayor expansion of the retardation $t-r/c$, and does not integrate over time (hence the adjective "instantaneous"); see Ref. [41] for the proof and more details about the iterated Poisson operator in Eq. (2.5).

The point, proved in the appendix of Paper I, is that the second term in Eq. (2.4) always contributes to *integral* post-Newtonian approximations, and thus can be safely ignored when looking at the half-integral approximations. We shall check in Appendix B below that the proof of Paper I is still applicable to the extended calculation performed here. Thus all the effects we are looking for come from the first term in Eq. (2.4), which is a homogeneous solution of the wave equation of the type retarded minus advanced and is parametrized by the function

$$G(u) = c\int_{-\infty}^{u}\mathrm{d}sR\left(\frac{u-s}{2},s\right). \tag{2.6}$$

Note that the retarded-minus-advanced solution is regular when $r\to 0$ and can therefore be directly extended by matching inside the source. The purpose is to compute the function G given the generic form of the source term S we need. As in Paper I we apply Eq. (2.6), together with Eq. (2.3), to source terms made of the requisite tails, that is, nonlocal in time (hereditary) terms having the form

$$S(r,t-r/c) = r^{B-k}\int_{1}^{+\infty}\mathrm{d}xQ_m(x)F(t-rx/c), \tag{2.7}$$

where F denotes some time derivative of a multipole moment, k and m are integers and $Q_m(x)$ is the Legendre function of the second kind, with branch cut from $-\infty$ to 1, explicitly given in terms of the usual Legendre polynomial $P_m(x)$ by

$$Q_m(x) = \frac{1}{2}P_m(x)\ln\left(\frac{x+1}{x-1}\right) - \sum_{j=1}^{m}\frac{1}{j}P_{m-j}(x)P_{j-1}(x). \tag{2.8}$$

Besides the hereditary source terms (2.7) we need also to include the case of instantaneous (nontail) terms of the type $S(r,t-r/c) = r^{B-k}F(t-r/c)$, but these are immediately deduced from the hereditary case (2.7) by replacing formally $Q_m(x)$ by the truncated delta function defined by $\delta_+(x-1) = Y(x-1)\delta(x-1)$, where Y and δ are the usual Heaviside and delta functions. Hence we can handle all the terms given for completeness in Appendix B.

Note that we systematically include inside the source term (2.7) a regularization factor r^B, where B is a complex parameter destined to tend to zero at the end of the calculation. The presence of this factor ensures, when the real part $\Re(B)$ is large enough, that the source term tends sufficiently rapidly toward zero when $r\to 0$, so the applicability conditions of the integration formulas (2.2) and (2.4) are fulfilled (see Refs. [39,41]). From the initial domain of the complex plane where $\Re(B)$ is large enough, we extend the validity of the formulas by analytic continuation to any complex B-values except isolated poles at integer values of B.

Plugging the source term (2.7) into Eq. (2.3), and then substituting (2.3) into Eq. (2.6), we obtained in Paper I a more tractable expression of the function G that parametrizes the term of interest to us in Eq. (2.4), namely

$$G(u) = c^{B+\ell-k+3}C_{k,\ell,m}(B)\int_0^{+\infty}\mathrm{d}\tau\tau^B F^{(k-\ell-2)}(u-\tau). \tag{2.9}$$

Always implicit in expressions such as Eq. (2.9) is that we perform the Laurent expansion of the result when $B\to 0$ and then pick up the finite part of that expansion, i.e. the coefficient of the zeroth power of B. Depending on the relative values of k and ℓ (namely the power of $1/r$ and the multipole order of the term in question), the function F in Eq. (2.9) will appear either multi time-differentiated or multi time-integrated, which we indicate in both cases by the superscript (p) where $p = k-\ell-2$ can be positive or negative; the formula (2.9) is valid in either case. The B-dependent coefficient $C_{k,\ell,m}$ in Eq. (2.9) reads

$$\begin{aligned}C_{k,\ell,m}(B) = {}& \frac{2^\ell}{\ell!}\frac{\Gamma(B-k+\ell+3)}{\Gamma(B+1)}\\ &\times\int_0^{+\infty}\mathrm{d}yQ_m(1+y)\\ &\times\int_0^1\mathrm{d}z\frac{z^{B-k-\ell+1}(1-z)^\ell}{(2+yz)^{B-k+\ell+3}},\end{aligned} \tag{2.10}$$

where Γ is the usual Eulerian function; see Paper I for more details. An alternative form of Eq. (2.10), also derived in Paper I, is

$$\begin{aligned}C_{k,\ell,m}(B) = {}& \frac{\Gamma(B-k-\ell+2)}{2\Gamma(B+1)}\\ &\times\sum_{i=0}^{\ell}\frac{(\ell+i)!}{i!(\ell-i)!}\frac{\Gamma(B-k+\ell+3)}{\Gamma(B-k+i+3)}\\ &\times\int_0^{+\infty}\mathrm{d}y\left(\frac{y}{2}\right)^i\frac{Q_m(1+y)}{(2+y)^{B-k+2}}.\end{aligned} \tag{2.11}$$

In order to control the tails present in the function G, and which are responsible for the half-integral post-Newtonian approximations (Paper I), we need to control the pole parts when $B\to 0$ of the expressions (2.10) or (2.11); see Eqs. (4.10)–(4.12) in Paper I. In particular, when only simple poles $\propto 1/B$ appear which will always be the case in the present paper, the tail part of the function G is given by

$$G^{\rm tail}(u) = c^{\ell-k+3}\alpha^{(-1)}_{k,\ell,m}\int_0^{+\infty}\mathrm{d}\tau\ln\tau F^{(k-\ell-2)}(u-\tau), \tag{2.12}$$

where $\alpha^{(-1)}_{k,\ell,m}$ denotes the residue (i.e. coefficient of $1/B$) in the Laurent expansion of the coefficient $C_{k,\ell,m}(B)$ when $B\to 0$. The residue can be obtained either by carefully expanding Eqs. (2.10) or (2.11) when $B\to 0$ as was done in Paper I, or by using a powerful alternative method, described in Appendix A, which is especially tuned to pick up directly and rapidly the required pole parts.

The tail integral (2.12) involves as usual a logarithmic kernel. Note that we keep the argument of the logarithm without a constant to adimensionalize it, e.g. $\ln(\tau/P)$, because any constant P will yield an instantaneous (nontail) term that is safely ignored here.

III. TAILS OF TAILS IN THE NEAR ZONE

A. Expressions in harmonic coordinates

A straightforward extension of the analysis of Paper I (see Sec. II therein) shows that in order to control the half-integral post-Newtonian coefficients up to next-to-next-to-leading order, namely 2PN beyond the leading-order 5.5PN coefficient obtained in Paper I, we need to compute the tails of tails associated with the mass-type quadrupole, octupole and hexadecapole moments, and with the current-type quadrupole and octupole moments. In the notation of Paper I, this means that we have to take into account the multipole interactions $M\times M\times I_{ij}$ (this one was sufficient for Paper I), $M\times M\times I_{ijk}$ and $M\times M\times I_{ijkl}$ for mass moments, as well as $M\times M\times J_{ij}$ and $M\times M\times J_{ijk}$ for current moments. As will be discussed in Sec. IV, those interactions represent only the "seeds" for a subsequent post-Newtonian iteration, formally involving higher nonlinear multipole interactions.

For all the seed multipole interactions we only need the functions G parametrizing the regular retarded-minus-advanced homogeneous solutions in Eq. (2.4). They are obtained from applying Eqs. (2.9)–(2.11) to each one of the source terms corresponding to these multipole interactions. The computation is straightforward, and for completeness we present in Appendix B the complete expressions of the required source terms, extending Eqs. (3.4)–(3.5) of Paper I. Typically all the coefficients in Eqs. (B3)–(B12) of Appendix B contribute to the final results. To ease the notation we use the following shorthand for an elementary monopolar retarded-minus-advanced homogeneous wave,

$$\{G(t)\}\equiv\frac{G(t-r/c)-G(t+r/c)}{r}. \tag{3.1}$$

Corresponding multipolar retarded-minus-advanced waves are obtained by applying STF partial space multiderivative operators $\hat{\partial}_L$ (with multipolarity ℓ). The near-zone expansion when $r\to 0$ of such multipolar waves is given by the Taylor expansion as

$$\overline{\hat{\partial}_L\{G(t)\}} = -2\sum_{k=0}^{+\infty}\frac{\hat{x}_L r^{2k}}{(2k)!!(2k+2\ell+1)!!}\frac{G^{(2k+2\ell+1)}(t)}{c^{2k+2\ell+1}}. \tag{3.2}$$

Extending Eqs. (5.1) of Paper I, we present the multipolar tail-of-tail interactions corresponding to the first term of Eq. (2.4), for each of the components of the gothic metric deviation $h^{\mu\nu}\equiv\sqrt{-g}g^{\mu\nu}-\eta^{\mu\nu}$ in harmonic gauge, such that $\partial_\nu h^{\mu\nu}=0$. All these contributions are built from the source terms given in Eqs. (B3)–(B12).

(i) Mass quadrupole moment:

$$(h^{00})_{M\times M\times I_{ij}} = \frac{116}{21}\frac{G^3M^2}{c^8}\int_0^{+\infty}\mathrm{d}\tau\ln\tau\partial_{ab}\{I^{(3)}_{ab}(t-\tau)\}, \tag{3.3a}$$

$$\begin{aligned}(h^{0i})_{M\times M\times I_{ij}} = {}& \frac{4}{105}\frac{G^3M^2}{c^7}\int_0^{+\infty}\mathrm{d}\tau\ln\tau\hat{\partial}_{iab}\{I^{(2)}_{ab}(t-\tau)\}\\ &-\frac{416}{75}\frac{G^3M^2}{c^9}\int_0^{+\infty}\mathrm{d}\tau\ln\tau\partial_a\{I^{(4)}_{ia}(t-\tau)\},\end{aligned} \tag{3.3b}$$

$$\begin{aligned}(h^{ij})_{M\times M\times I_{ij}} = {}& -\frac{32}{21}\frac{G^3M^2}{c^8}\int_0^{+\infty}\mathrm{d}\tau\ln\tau\delta_{ij}\partial_{ab}\{I^{(3)}_{ab}(t-\tau)\}\\ &+\frac{104}{35}\frac{G^3M^2}{c^8}\int_0^{+\infty}\mathrm{d}\tau\ln\tau\hat{\partial}_{a(i}\{I^{(3)}_{j)a}(t-\tau)\}\\ &+\frac{76}{15}\frac{G^3M^2}{c^{10}}\int_0^{+\infty}\mathrm{d}\tau\ln\tau\{I^{(5)}_{ij}(t-\tau)\}.\end{aligned} \tag{3.3c}$$

(ii) Mass octupole:

$$(h^{00})_{M\times M\times I_{ijk}} = -\frac{328}{315}\frac{G^3M^2}{c^8}\int_0^{+\infty}\mathrm{d}\tau\ln\tau\partial_{abc}\{I^{(3)}_{abc}(t-\tau)\}, \tag{3.4a}$$

$$\begin{aligned}(h^{0i})_{M\times M\times I_{ijk}} = {}& -\frac{2}{315}\frac{G^3M^2}{c^7}\int_0^{+\infty}\mathrm{d}\tau\ln\tau\hat{\partial}_{iabc}\{I^{(2)}_{abc}(t-\tau)\}\\ &+\frac{256}{245}\frac{G^3M^2}{c^9}\int_0^{+\infty}\mathrm{d}\tau\ln\tau\partial_{ab}\{I^{(4)}_{iab}(t-\tau)\},\end{aligned} \tag{3.4b}$$

$$(h^{ij})_{M\times M\times I_{ijk}} = \frac{8}{35}\frac{G^3M^2}{c^8}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \delta_{ij}\partial_{abc}\{I^{(3)}_{abc}(t-\tau)\} - \frac{4}{9}\frac{G^3M^2}{c^8}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \hat{\partial}_{ab(i}\{I^{(3)}_{j)ab}(t-\tau)\} - \frac{316}{315}\frac{G^3M^2}{c^{10}}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \partial_a\{I^{(5)}_{ija}(t-\tau)\}. \tag{3.4c}$$

(iii) Mass hexadecapole:

$$(h^{00})_{M\times M\times I_{ijkl}} = \frac{1898}{10395}\frac{G^3M^2}{c^8}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \partial_{abcd}\{I^{(3)}_{abcd}(t-\tau)\}, \tag{3.5a}$$

$$(h^{0i})_{M\times M\times I_{ijkl}} = \frac{1}{1155}\frac{G^3M^2}{c^7}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \hat{\partial}_{iabcd}\{I^{(2)}_{abcd}(t-\tau)\} - \frac{173}{945}\frac{G^3M^2}{c^9}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \partial_{abc}\{I^{(4)}_{iabc}(t-\tau)\}, \tag{3.5b}$$

$$(h^{ij})_{M\times M\times I_{ijkl}} = -\frac{23}{693}\frac{G^3M^2}{c^8}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \delta_{ij}\partial_{abcd}\{I^{(3)}_{abcd}(t-\tau)\} + \frac{32}{495}\frac{G^3M^2}{c^8}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \hat{\partial}_{abc(i}\{I^{(3)}_{j)abc}(t-\tau)\} + \frac{169}{945}\frac{G^3M^2}{c^{10}}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \partial_{ab}\{I^{(5)}_{ijab}(t-\tau)\}. \tag{3.5c}$$

(iv) Current quadrupole[4]:

$$(h^{00})_{M\times M\times J_{ij}} = 0, \tag{3.6a}$$

$$(h^{0i})_{M\times M\times J_{ij}} = \frac{296}{105}\frac{G^3M^2}{c^9}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \varepsilon_{iab}\partial_{bc}\{J^{(3)}_{ac}(t-\tau)\}, \tag{3.6b}$$

$$(h^{ij})_{M\times M\times J_{ij}} = -\frac{64}{315}\frac{G^3M^2}{c^8}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \varepsilon_{ab(i}\hat{\partial}_{j)bc}\{J^{(2)}_{ac}(t-\tau)\} - \frac{1232}{225}\frac{G^3M^2}{c^{10}}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \varepsilon_{ab(i}\partial_{\underline{b}}\{J^{(4)}_{j)a}(t-\tau)\}. \tag{3.6c}$$

(v) Current octupole:

$$(h^{00})_{M\times M\times J_{ijk}} = 0, \tag{3.7a}$$

$$(h^{0i})_{M\times M\times J_{ijk}} = -\frac{68}{105}\frac{G^3M^2}{c^9}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \varepsilon_{iab}\partial_{bcd}\{J^{(3)}_{acd}(t-\tau)\}, \tag{3.7b}$$

$$(h^{ij})_{M\times M\times J_{ijk}} = \frac{2}{35}\frac{G^3M^2}{c^8}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \varepsilon_{ab(i}\hat{\partial}_{j)bcd}\{J^{(2)}_{acd}(t-\tau)\} + \frac{922}{735}\frac{G^3M^2}{c^{10}}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \varepsilon_{ab(i}\partial_{\underline{bc}}\{J^{(4)}_{j)ac}(t-\tau)\}. \tag{3.7c}$$

B. Application of a gauge transformation

As noticed in Paper I the tail-of-tail term $M\times M\times I_{ij}$ given by Eqs. (3.3) is to be iterated at higher nonlinear order as there are some post-Newtonian terms which contribute at the same level coming from higher nonlinear iterations. However it was found that the details of that nonlinear iteration depend on the adopted coordinate system. In Paper I two computations of the 5.5PN coefficient were made, one in the standard harmonic coordinate system, based on the previous expressions (3.3), and one in an alternative coordinate system in which the 5.5PN terms in the $0i$ and ij components of the metric are "transferred" to the 00 component at that order. This alternative coordinate system has the great advantage that it considerably simplifies the subsequent nonlinear iteration. Actually, it was found in Paper I that at 5.5PN order in this coordinate system there is no need to perform the nonlinear iteration. Such a coordinate system is analogous to the Burke and Thorne coordinate system [42,43] (see also [44]), in which the complete radiation reaction force at the 2.5PN order is linear, with nonlinear contributions arising only at higher post-Newtonian orders.

In the present paper we shall systematically work in the alternative nonharmonic coordinate system so designed that it minimizes (but, at such high 7.5PN order, does not suppress) the need for controlling nonlinear contributions. Even in that optimized gauge we shall find that the nonlinear contributions are numerous and require two

[4]Underlined indices mean that they should be excluded from the symmetrization $T_{(ij)} = \frac{1}{2}(T_{ij}+T_{ji})$.

iterations. We did not attempt to perform these nonlinear iterations in harmonic coordinates. Since the redshift factor we compute in fine is gauge invariant we are allowed to use whatever coordinate system we like. Thus we proceed with introducing appropriate gauge transformation vectors η^μ to be applied to each of the multipolar pieces presented in Sec. III A. The complete gauge transformation is of course the sum of each of the separate multipolar pieces. At leading 5.5PN order the mass quadrupole piece agrees with Eqs. (5.11) of Paper I, except that here we do not yet focus our attention on the conservative part of the dynamics; a split between conservative and dissipative parts will be made at a later stage; see Eqs. (4.43). Note also that the above gauge vectors generalize those of Paper I not only because they involve more multipole interactions but also because they include all post-Newtonian terms, i.e. complete series expansions such as Eq. (3.2).

(i) Mass quadrupole:

$$(\eta^0)_{M\times M\times I_{ij}} = \frac{77}{15}\frac{G^3M^2}{c^7}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \partial_{ab}\{I^{(2)}_{ab}(t-\tau)\}, \tag{3.8a}$$

$$(\eta^i)_{M\times M\times I_{ij}} = -\frac{107}{3}\frac{G^3M^2}{c^6}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \hat{\partial}_{iab}\{I^{(1)}_{ab}(t-\tau)\} + \frac{38}{5}\frac{G^3M^2}{c^8}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \partial_{a}\{I^{(3)}_{ia}(t-\tau)\}. \tag{3.8b}$$

(ii) Mass octupole:

$$(\eta^0)_{M\times M\times I_{ijk}} = -\frac{461}{945}\frac{G^3M^2}{c^7}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \partial_{abc}\{I^{(2)}_{abc}(t-\tau)\}, \tag{3.9a}$$

$$(\eta^i)_{M\times M\times I_{ijk}} = \frac{13}{3}\frac{G^3M^2}{c^6}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \hat{\partial}_{iabc}\{I^{(1)}_{abc}(t-\tau)\} - \frac{79}{63}\frac{G^3M^2}{c^8}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \partial_{ab}\{I^{(3)}_{iab}(t-\tau)\}. \tag{3.9b}$$

(iii) Mass hexadecapole:

$$(\eta^0)_{M\times M\times I_{ijkl}} = \frac{29}{504}\frac{G^3M^2}{c^7}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \partial_{abcd}\{I^{(2)}_{abcd}(t-\tau)\}, \tag{3.10a}$$

$$(\eta^i)_{M\times M\times I_{ijkl}} = -\frac{1571}{2520}\frac{G^3M^2}{c^6}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \hat{\partial}_{iabcd}\{I^{(1)}_{abcd}(t-\tau)\} + \frac{169}{810}\frac{G^3M^2}{c^8}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \partial_{abc}\{I^{(3)}_{iabc}(t-\tau)\}. \tag{3.10b}$$

(iv) Current quadrupole:

$$(\eta^0)_{M\times M\times J_{ij}} = 0, \tag{3.11a}$$

$$(\eta^i)_{M\times M\times J_{ij}} = -\frac{616}{45}\frac{G^3M^2}{c^8}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \varepsilon_{iab}\partial_{bc}\{J^{(2)}_{ac}(t-\tau)\}. \tag{3.11b}$$

(v) Current octupole:

$$(\eta^0)_{M\times M\times J_{ijk}} = 0, \tag{3.12a}$$

$$(\eta^i)_{M\times M\times J_{ijk}} = \frac{461}{210}\frac{G^3M^2}{c^8}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \varepsilon_{iab}\partial_{bcd}\{J^{(2)}_{acd}(t-\tau)\}. \tag{3.12b}$$

Applying the latter linear gauge transformations we obtain new expressions for the gothic metric coefficients, say $h'^{\mu\nu}$. Our convention is that (for each multipole component)

$$h'^{\mu\nu} = h^{\mu\nu} - \partial^\mu\eta^\nu - \partial^\nu\eta^\mu + \eta^{\mu\nu}\partial_\rho\eta^\rho. \tag{3.13}$$

The nice property of the metric in the new gauge is that the number ℓ of STF spatial derivatives $\hat{\partial}_L$ for each multipole is maximal, and equal to $\ell = m + s$ for mass moments and $\ell = m + s - 1$ for current moments, where m is the multipolarity of the multipole moment in question (i.e. I_M or J_M) and s is the number of spatial indices in the gothic metric (i.e. $s = 0, 1, 2$ according to whether $\mu\nu = 00, 0i, ij$). From Eq. (3.2) we see that maximizing the number of STF derivatives means pushing to the maximum the leading PN order, and therefore minimizing the need of nonlinear iterations at a given PN level.

(i) Mass quadrupole:

$$(h'^{00})_{M\times M\times I_{ij}} = \frac{856}{35}\frac{G^3M^2}{c^8}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \partial_{ab}\{I^{(3)}_{ab}(t-\tau)\}, \tag{3.14a}$$

$$(h'^{0i})_{M\times M\times I_{ij}} = -\frac{856}{21}\frac{G^3M^2}{c^7}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \hat{\partial}_{iab}\{I^{(2)}_{ab}(t-\tau)\}, \tag{3.14b}$$

$$(h'^{ij})_{M\times M\times I_{ij}} = \frac{214}{3}\frac{G^3M^2}{c^6}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \hat{\partial}_{ijab}\{I^{(1)}_{ab}(t-\tau)\}. \tag{3.14c}$$

(ii) Mass octupole:

$$(h'^{00})_{M\times M\times I_{ijk}} = -\frac{520}{189}\frac{G^3M^2}{c^8}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \partial_{abc}\{I^{(3)}_{abc}(t-\tau)\}, \tag{3.15a}$$

$$(h'^{0i})_{M\times M\times I_{ijk}} = \frac{130}{27}\frac{G^3M^2}{c^7}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \hat{\partial}_{iabc}\{I^{(2)}_{abc}(t-\tau)\}, \tag{3.15b}$$

$$(h'^{ij})_{M\times M\times I_{ijk}} = -\frac{26}{3}\frac{G^3M^2}{c^6}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \hat{\partial}_{ijabc}\{I^{(1)}_{abc}(t-\tau)\}. \tag{3.15c}$$

(iii) Mass hexadecapole:

$$(h'^{00})_{M\times M\times I_{ijkl}} = \frac{1571}{4158}\frac{G^3M^2}{c^8}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \partial_{abcd}\{I^{(3)}_{abcd}(t-\tau)\}, \tag{3.16a}$$

$$(h'^{0i})_{M\times M\times I_{ijkl}} = -\frac{1571}{2310}\frac{G^3M^2}{c^7}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \hat{\partial}_{iabcd}\{I^{(2)}_{abcd}(t-\tau)\}, \tag{3.16b}$$

$$(h'^{ij})_{M\times M\times I_{ijkl}} = \frac{1571}{1260}\frac{G^3M^2}{c^6}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \hat{\partial}_{ijabcd}\{I^{(1)}_{abcd}(t-\tau)\}. \tag{3.16c}$$

(iv) Current quadrupole:

$$(h'^{00})_{M\times M\times J_{ij}} = 0, \tag{3.17a}$$

$$(h'^{0i})_{M\times M\times J_{ij}} = -\frac{3424}{315}\frac{G^3M^2}{c^9}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \varepsilon_{iab}\partial_{bc}\{J^{(3)}_{ac}(t-\tau)\}, \tag{3.17b}$$

$$(h'^{ij})_{M\times M\times J_{ij}} = \frac{1712}{63}\frac{G^3M^2}{c^8}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \varepsilon_{ab(i}\hat{\partial}_{j)bc}\{J^{(2)}_{ac}(t-\tau)\}. \tag{3.17c}$$

(v) Current octupole:

$$(h'^{00})_{M\times M\times J_{ijk}} = 0, \tag{3.18a}$$

$$(h'^{0i})_{M\times M\times J_{ijk}} = \frac{65}{42}\frac{G^3M^2}{c^9}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \varepsilon_{iab}\partial_{bcd}\{J^{(3)}_{acd}(t-\tau)\}, \tag{3.18b}$$

$$(h'^{ij})_{M\times M\times J_{ijk}} = -\frac{13}{3}\frac{G^3M^2}{c^8}\int_0^{+\infty} \mathrm{d}\tau \ln\tau \varepsilon_{ab(i}\hat{\partial}_{j)bcd}\{J^{(2)}_{acd}(t-\tau)\}. \tag{3.18c}$$

Notice that $h'^{ii}_{(1)} = 0$ for all these pieces, which is a nice feature of the new gauge, shared in fact with the harmonic gauge. Recall that expressions (3.14)–(3.18) are regular inside the source and will be valid as they stand at the location of the particles in a binary system.

IV. POST-NEWTONIAN ITERATION OF TAILS OF TAILS

A. Setting up the iteration

As mentioned above, we found in Paper I that in harmonic coordinates the computation of the 5.5PN coefficient requires the control of one nonlinear PN iteration, but that no nonlinear iteration is needed in the alternative nonharmonic gauge. To extend the result up to 7.5PN order, our rationale here is to systematically use the simpler nonharmonic gauge in which the metric components are given by Eqs. (3.14)–(3.18).

In the iteration process we shall have to couple the tail-of-tail pieces (3.14)–(3.18) with the lower order 1PN metric. Since the choice of nonharmonic gauge we have made above affects only the higher order tail-of-tail parts of the metric, we can take for the 1PN metric the standard form in harmonic coordinates, given by

$$h^{00} = -\frac{4}{c^2}V - \frac{2}{c^4}(\hat{W} + 4V^2) + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{4.1a}$$

$$h^{0i} = -\frac{4}{c^3}V_i + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{4.1b}$$

$$h^{ij} = -\frac{4}{c^4}\left(\hat{W}_{ij} - \frac{1}{2}\delta_{ij}\hat{W}\right) + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{4.1c}$$

where we follow our usual notation for appropriate metric potentials V, V_i, $\hat{W}_{ij}$ and $\hat{W} = \hat{W}_{kk}$, defined in Sec. (5.3) of Ref. [32] for general post-Newtonian sources. We then specialize these potentials to point particle binary sources. Denoting the masses by m_A ($A = 1, 2$), the trajectories and velocities by $y^i_A(t)$ and $v^i_A(t) = \mathrm{d}y^i_A(t)/\mathrm{d}t$, the distances to the field point by $r_A = |\mathbf{x} - \mathbf{y}_A|$ and the separation by $r_{12} = |\mathbf{y}_1 - \mathbf{y}_2|$, we have

$$V = U + \frac{1}{c^2}\partial_t^2 U_2 + \mathcal{O}\left(\frac{1}{c^3}\right), \tag{4.2a}$$

$$V_i = U_i + \mathcal{O}\left(\frac{1}{c^2}\right), \tag{4.2b}$$

$$\hat{W}_{ij} = U_{ij} - \delta_{ij} U_{kk} - \frac{G^2 m_1^2}{8}\left[\partial_{ij} \ln r_1 + \frac{\delta^{ij}}{r_1^2}\right] - \frac{G^2 m_2^2}{8}\left[\partial_{ij} \ln r_2 + \frac{\delta^{ij}}{r_2^2}\right] - 2G^2 m_1 m_2 \frac{\partial^2 g}{\partial y_1^{(i} \partial y_2^{j)}} + \mathcal{O}\left(\frac{1}{c}\right). \tag{4.2c}$$

Here U, U_i and U_{ij} refer to the compact-support parts of the potentials that are given (consistently with the approximation) explicitly by

$$U = \frac{G\tilde{\mu}_1}{r_1} + \frac{G\tilde{\mu}_2}{r_2}, \tag{4.3a}$$

$$U_2 = \frac{G\tilde{\mu}_1}{2} r_1 + \frac{G\tilde{\mu}_2}{2} r_2, \tag{4.3b}$$

$$U_i = \frac{Gm_1}{r_1} v_1^i + \frac{Gm_2}{r_2} v_2^i, \tag{4.3c}$$

$$U_{ij} = \frac{Gm_1}{r_1} v_1^i v_1^j + \frac{Gm_2}{r_2} v_2^i v_2^j. \tag{4.3d}$$

The potential U is 1PN accurate and we have introduced the effective time-dependent masses at 1PN order (which are pure functions of time),

$$\tilde{\mu}_1 = m_1\left[1 - \frac{Gm_2}{c^2 r_{12}} + \frac{3}{2}\frac{v_1^2}{c^2}\right], \tag{4.4}$$

and $\tilde{\mu}_2$ obtained by exchanging the particle labels. Note that the potential U_2 so defined is the "superpotential" of U, in the sense that

$$\Delta U_2 = U. \tag{4.5}$$

Later, we shall systematically make use of the notion of high-order superpotentials. Finally the nonlinear interaction term in (4.2c) is expressed by means of the well-known function [45]

$$g = \ln(r_1 + r_2 + r_{12}), \tag{4.6}$$

which is the superpotential of $1/(r_1 r_2)$, i.e. $\Delta g = \frac{1}{r_1 r_2}$ in the sense of distributions. Later we shall introduce the superpotential of g itself. In Eq. (4.2c) the function g is differentiated with respect to the two source points y_A^i as indicated.

The most important problem we face is the mass quadrupole case, which we shall need to iterate two times. We need to control the covariant metric components g_{00}, g_{0i} and g_{ij} up to order 7.5PN, which means c^{-17}, c^{-16} and c^{-15} included, i.e. up to remainders $\mathcal{O}(c^{-19})$, $\mathcal{O}(c^{-18})$ and $\mathcal{O}(c^{-17})$ respectively. We first write the metric components in the new gauge obtained in Eqs. (3.14)–(3.18) up to the required order, with the help of Eq. (3.2). For convenience we simply denote e.g. $\delta h^{\mu\nu}_{(1)} = (h'^{\mu\nu})_{M\times M\times I_{ij}}$, forgetting about the prime indicating the new gauge and also about the type of multipole interaction. However we call this piece $\delta h^{\mu\nu}_{(1)}$ because we shall eventually obtain iterated and twice-iterated contributions $\delta h^{\mu\nu}_{(2)}$ and $\delta h^{\mu\nu}_{(3)}$.

(i) Mass quadupole:

$$\delta h^{00}_{(1)} = -\frac{1712}{525}\frac{G^3 M^2}{c^{13}} \hat{x}_{ab} \int_0^{+\infty} \mathrm{d}\tau \ln\tau \left[I^{(8)}_{ab}(t-\tau) + \frac{r^2}{14c^2} I^{(10)}_{ab}(t-\tau) + \frac{r^4}{504c^4} I^{(12)}_{ab}(t-\tau)\right] + \mathcal{O}\left(\frac{1}{c^{19}}\right), \tag{4.7a}$$

$$\delta h^{0i}_{(1)} = \frac{1712}{2205}\frac{G^3 M^2}{c^{14}} \hat{x}_{iab} \int_0^{+\infty} \mathrm{d}\tau \ln\tau \left[I^{(9)}_{ab}(t-\tau) + \frac{r^2}{18c^2} I^{(11)}_{ab}(t-\tau)\right] + \mathcal{O}\left(\frac{1}{c^{18}}\right), \tag{4.7b}$$

$$\delta h^{ij}_{(1)} = -\frac{428}{2835}\frac{G^3 M^2}{c^{15}} \hat{x}_{ijab} \int_0^{+\infty} \mathrm{d}\tau \ln\tau I^{(10)}_{ab}(t-\tau) + \mathcal{O}\left(\frac{1}{c^{17}}\right). \tag{4.7c}$$

(ii) Mass octupole:

$$\delta h^{00}_{(1)} = \frac{208}{3969}\frac{G^3 M^2}{c^{15}} \hat{x}_{abc} \int_0^{+\infty} \mathrm{d}\tau \ln\tau \left[I^{(10)}_{abc}(t-\tau) + \frac{r^2}{18c^2} I^{(12)}_{abc}(t-\tau)\right] + \mathcal{O}\left(\frac{1}{c^{19}}\right), \tag{4.8a}$$

$$\delta h^{0i}_{(1)} = -\frac{52}{5103}\frac{G^3 M^2}{c^{16}} \hat{x}_{iabc} \int_0^{+\infty} \mathrm{d}\tau \ln\tau I^{(11)}_{abc}(t-\tau) + \mathcal{O}\left(\frac{1}{c^{18}}\right), \tag{4.8b}$$

$$\delta h^{ij}_{(1)} = \mathcal{O}\left(\frac{1}{c^{17}}\right). \tag{4.8c}$$

(iii) Mass hexadecapole:

$$\delta h^{00}_{(1)} = -\frac{1571}{1964655}\frac{G^3M^2}{c^{17}}\hat{x}_{abcd}\int_0^{+\infty} \mathrm{d}\tau \ln \tau I^{(12)}_{abcd}(t-\tau) + \mathcal{O}\left(\frac{1}{c^{19}}\right), \tag{4.9a}$$

$$\delta h^{0i}_{(1)} = \mathcal{O}\left(\frac{1}{c^{18}}\right), \tag{4.9b}$$

$$\delta h^{ij}_{(1)} = \mathcal{O}\left(\frac{1}{c^{19}}\right). \tag{4.9c}$$

(iv) Current quadrupole:

$$\delta h^{00}_{(1)} = 0, \tag{4.10a}$$

$$\delta h^{0i}_{(1)} = \frac{6848}{4725}\frac{G^3M^2}{c^{14}}\varepsilon_{iab}\hat{x}_{bc}\int_0^{+\infty} \mathrm{d}\tau \ln \tau \left[J^{(8)}_{ac}(t-\tau) + \frac{r^2}{14c^2}J^{(10)}_{ac}(t-\tau)\right] + \mathcal{O}\left(\frac{1}{c^{18}}\right), \tag{4.10b}$$

$$\delta h^{ij}_{(1)} = -\frac{3424}{6615}\frac{G^3M^2}{c^{15}}\varepsilon_{ab(i}\hat{x}_{j)bc}\int_0^{+\infty} \mathrm{d}\tau \ln \tau J^{(9)}_{ac}(t-\tau) + \mathcal{O}\left(\frac{1}{c^{17}}\right). \tag{4.10c}$$

(v) Current octupole:

$$\delta h^{00}_{(1)} = 0, \tag{4.11a}$$

$$\delta h^{0i}_{(1)} = -\frac{13}{441}\frac{G^3M^2}{c^{16}}\varepsilon_{iab}\hat{x}_{bcd}\int_0^{+\infty} \mathrm{d}\tau \ln \tau J^{(10)}_{acd}(t-\tau) + \mathcal{O}\left(\frac{1}{c^{18}}\right), \tag{4.11b}$$

$$\delta h^{ij}_{(1)} = \mathcal{O}\left(\frac{1}{c^{17}}\right). \tag{4.11c}$$

B. Quadratic iteration

At quadratic nonlinear order we have to solve the equation

$$\Box h^{\mu\nu}_{(2)} = N^{\mu\nu}_{(2)}, \tag{4.12}$$

where the source term $N^{\mu\nu}_{(2)}$ is made of quadratic products of derivatives of $h^{\mu\nu}_{(1)}$, symbolically written as $\sim\partial h_{(1)}\partial h_{(1)}$ and $\sim h_{(1)}\partial^2 h_{(1)}$. Here $h^{\mu\nu}_{(1)}$ is composed by the 1PN metric (4.1) augmented at high orders by all the previous tail-of-tail pieces. Note that Eq. (4.12) is valid in the new gauge but with the assumption that at the next nonlinear order the harmonic gauge condition is satisfied, i.e. $\partial_\nu h^{\mu\nu}_{(2)} = 0$, and later, at still higher order, we shall assume the same, $\partial_\nu h^{\mu\nu}_{(3)} = 0$ (such choices are simply a matter of convenience). The quadratic terms we need for the present iteration, consistent with the order 7.5PN, are

$$\begin{aligned} N^{00}_{(2)} + N^{ii}_{(2)} = {} & -h^{00}_{(1)}\partial_{00}h^{00}_{(1)} - 2h^{0i}_{(1)}\partial_{0i}h^{00}_{(1)} - h^{ij}_{(1)}\partial_{ij}h^{00}_{(1)} \\ & - \partial_i h^{00}_{(1)}\partial_i h^{00}_{(1)} - \frac{1}{2}(\partial_0 h^{00}_{(1)})^2 + 2\partial_0 h^{0i}_{(1)}\partial_0 h^{0i}_{(1)} \\ & + 4\partial_0 h^{ij}_{(1)}\partial_i h^{0j}_{(1)} + 2\partial_i h^{0j}_{(1)}\partial_j h^{0i}_{(1)} + \partial_i h^{jk}_{(1)}\partial_i h^{jk}_{(1)}, \end{aligned} \tag{4.13a}$$

$$N^{0i}_{(2)} = \frac{3}{4}\partial_0 h^{00}_{(1)}\partial_i h^{00}_{(1)} + \partial_j h^{00}_{(1)}\partial_i h^{0j}_{(1)} - \partial_j h^{00}_{(1)}\partial_j h^{0i}_{(1)}, \tag{4.13b}$$

$$N^{ij}_{(2)} = \frac{1}{4}\partial_i h^{00}_{(1)}\partial_j h^{00}_{(1)} - \frac{1}{8}\delta_{ij}\partial_k h^{00}_{(1)}\partial_k h^{00}_{(1)}. \tag{4.13c}$$

Since we are ultimately interested in the covariant metric components $g_{\mu\nu}$ we are considering the combination $00+ii$ of gothic metric components which appears dominantly into g_{00}.

We now replace in (4.13) the gothic metric by its explicit form which reduces up to 1PN order to Eqs. (4.1) and involves all the tail-of-tail pieces $\delta h^{\mu\nu}_{(1)}$. Obviously the iterated quadratic tail-of-tail pieces will come from the cross products between the 1PN metric (4.1) and the linear tails of tails (4.7)–(4.11). When considering such cross products, we shall have to integrate typical terms whose general structure is $\hat{x}_L\phi$, where ϕ is any of the potentials appearing in Eqs. (4.1), and where $\hat{x}_L$ denotes a STF product of spatial vectors coming from Eqs. (4.7)–(4.11). Notice that the hereditary integrals therein are simply functions of time [see e.g. (4.24)], and essentially play a spectator role in the process, with the notable exception of that in the dominant term for which we have to consider a retardation at the relative 1PN order. In addition, because of the 1PN retardation in $\delta h^{\mu\nu}_{(1)}$, we shall have to integrate the slightly more complicated source term $r^2\hat{x}_L\phi$ [see e.g. Eq. (4.7a)]. Thus, the equations we have to solve are

$$\Delta\Psi_L = \hat{x}_L\phi, \tag{4.14a}$$

$$\Delta\Phi_L = r^2\hat{x}_L\phi. \tag{4.14b}$$

To solve them we adopt the method of superpotentials. Namely we introduce, given the potential ϕ, the hierarchy of its superpotentials denoted ϕ_{2k+2}, for any positive integer k, where $\phi_0 = \phi$ and

$$\Delta\phi_{2k+2} = \phi_{2k}. \tag{4.15}$$

We thus have $\Delta^k\phi_{2k} = \phi$. The explicit formulas for the solutions of Eqs. (4.14) (the first one being needed for $\ell = 4$, and the second one only for $\ell = 0, 1$) are

$$\Psi_L = \Delta^{-1}(\hat{x}_L\phi) = \sum_{k=0}^{\ell}\frac{(-2)^k\ell!}{(\ell-k)!}x^{\langle L-K}\partial_{K\rangle}\phi_{2k+2}, \tag{4.16a}$$

$$\begin{aligned}\Phi_L &= \Delta^{-1}(r^2\hat{x}_L\phi)\\ &= \sum_{k=0}^{\ell}\frac{(-2)^k\ell!}{(\ell-k)!}x^{\langle L-K}\partial_{K\rangle}\\ &\quad\times[r^2\phi_{2k+2} + 2(k+1)(2k+1)\\ &\quad\times\phi_{2k+4} - 4(k+1)x^i\partial_i\phi_{2k+4}].\end{aligned} \tag{4.16b}$$

These solutions are unique in the following sense. Suppose that ϕ admits an asymptotic expansion when $r\to\infty$ (with t fixed) on the set of basis functions $r^{\lambda-n}$, labeled by $n\in\mathbb{N}$ and where the maximal power is $\lambda\in\mathbb{R}\backslash\mathbb{N}$ (i.e. is not an integer). Then, for instance, the solution Ψ_L given by Eq. (4.16a) is the unique solution of Eq. (4.14a), valid in the sense of distribution theory [46], that admits an asymptotic expansion when $r\to\infty$ on the basis functions $r^{\lambda+\ell+2-n}$. Similarly Φ_L is the unique solution in the sense of distributions which admits an asymptotic expansion on the basis $r^{\lambda+\ell+4-n}$. The formulas (4.16) can be easily proved by induction. They can also be iterated if necessary; for instance we find by iterating i times the first one that

$$\begin{aligned}\Delta^{-i}\Psi_L &= \Delta^{-i-1}(\hat{x}_L\phi)\\ &= \sum_{k=0}^{\ell}\frac{(-2)^k(k+i)!\ell!}{k!i!(\ell-k)!}x^{\langle L-K}\partial_{K\rangle}\phi_{2k+2i+2}.\end{aligned} \tag{4.17}$$

Let us give an example of the applicability of those formulas. The 1PN compact-support potential U was defined by Eq. (4.3a), where we recall that the effective masses $\tilde{\mu}_A$ are mere functions of time. Now the hierarchy of superpotentials of U is given by

$$U_{2k} = \frac{1}{(2k)!}[G\tilde{\mu}_1 r_1^{2k-1} + G\tilde{\mu}_2 r_2^{2k-1}]. \tag{4.18}$$

For $k=1$ we recover the potential U_2 already met in Eq. (4.3b). Of course similar expressions apply for the other potentials U_i and U_{ij} in Eqs. (4.3). Taking only the leading-order cross term in the expression of the nonlinear source (4.13a) we find that we have to solve

$$\Box\Psi = x_i\partial_j V F_{ij}(t). \tag{4.19}$$

Here V is the retarded potential (4.2a) and F_{ij} is a certain function of time, which we shall define below to be the hereditary integral (4.24). Note that Eq. (4.19) is to be solved including the first-order retardation at 1PN order, which is simply done by using the symmetric propagator $\Box^{-1} = \Delta^{-1} + \frac{1}{c^2}\partial_t^2\Delta^{-2} + \mathcal{O}(c^{-2})$. Using then our elementary solution (4.16a) we get, up to 1PN relative order,

$$\begin{aligned}\Psi &= \Box^{-1}[x_i\partial_j V F_{ij}(t)]\\ &= (x_i\partial_j U_2 - 2\partial_{ij}U_4)F_{ij}\\ &\quad + \frac{1}{c^2}[x_i(2\partial_j\partial_t^2 U_4 F_{ij} + 2\partial_j\partial_t U_4 F^{(1)}_{ij} + \partial_j U_4 F^{(2)}_{ij})\\ &\quad - 6\partial_{ij}\partial_t^2 U_6 F_{ij} - 8\partial_{ij}\partial_t U_6 F^{(1)}_{ij} - 4\partial_{ij}U_6 F^{(2)}_{ij}]\\ &\quad + \mathcal{O}\left(\frac{1}{c^4}\right).\end{aligned} \tag{4.20}$$

Below we shall need not only the superpotentials of a compact-support potential like U, but also those of more complicated potentials such as $\hat{W}_{ij}$ defined by Eq. (4.2c). Its first order superpotential reads (to Newtonian order)

$$\begin{aligned}\hat{W}^{ij}_2 &= U^{ij}_2 - \delta^{ij}U^{kk}_2\\ &\quad - \frac{G^2m_1^2}{8}\left[\partial_{ij}\left(\frac{r_1^2}{6}\left(\ln r_1 - \frac{5}{6}\right)\right) + \delta^{ij}\ln r_1\right]\\ &\quad - \frac{G^2m_2^2}{8}\left[\partial_{ij}\left(\frac{r_2^2}{6}\left(\ln r_2 - \frac{5}{6}\right)\right) + \delta^{ij}\ln r_2\right]\\ &\quad - G^2m_1m_2\frac{\partial^2 f}{\partial y_1^{(i}\partial y_2^{j)}},\end{aligned} \tag{4.21a}$$

where we have used the superpotential of U_{ij} as well as the one of the function g of Eq. (4.6), namely $g_2 = f/2$ defined by

$$f = \frac{1}{3}\mathbf{r}_1 \cdot \mathbf{r}_2 \left[g - \frac{1}{3}\right] + \frac{1}{6}(r_1 r_{12} + r_2 r_{12} - r_1 r_2), \tag{4.22}$$

where $\mathbf{r}_A = \mathbf{x} - \mathbf{y}_A$, which satisfies $\Delta f = 2g$ in the sense of distributions (see e.g. Ref. [47]). A full hierarchy of higher superpotentials for the function g could be defined similarly. Note that the superpotentials of the noncompact potential U^2 are obtained thanks to the superpotentials of g (at Newtonian order say, i.e. assimilating $\tilde{\mu}_A$ to m_A):

$$U^2 = \frac{G^2 m_1^2}{r_1^2} + \frac{G^2 m_2^2}{r_2^2} + 2\frac{G^2 m_1 m_2}{r_1 r_2}, \tag{4.23a}$$

$$(U^2)_2 = G^2 m_1^2 \ln r_1 + G^2 m_2^2 \ln r_2 + 2G^2 m_1 m_2 g, \tag{4.23b}$$

$$(U^2)_4 = \frac{G^2 m_1^2}{6} r_1^2 \left(\ln r_1 - \frac{5}{6}\right) + \frac{G^2 m_2^2}{6} r_2^2 \left(\ln r_2 - \frac{5}{6}\right) + G^2 m_1 m_2 f. \tag{4.23c}$$

We now define as a convenient shorthand the following hereditary function of time appropriate for the mass quadrupole moment,

$$F_{ij}(t) = -\frac{1712}{525} G^3 M^2 \int_0^{+\infty} \mathrm{d}\tau \ln \tau I_{ij}^{(8)}(t-\tau), \tag{4.24}$$

and obtain the full expressions of $\delta h^{\mu\nu}_{(2)}$ up to the requested PN order as

$$\begin{aligned}
\delta h^{00}_{(2)} + \delta h^{ii}_{(2)} = {} & \frac{16}{c^{15}}(x_i \partial_j U_2 - 2\partial_{ij} U_4) F_{ij} \\
& + \frac{1}{c^{17}} \Big[16 x_i (2\partial_i \partial_t^2 U_4 F_{ij} + 2\partial_i \partial_t U_4 F_{ij}^{(1)} + \partial_i U_4 F_{ij}^{(2)}) \\
& + 16(-6\partial_{ij}\partial_t^2 U_6 F_{ij} - 8\partial_{ij}\partial_t U_6 F_{ij}^{(1)} - 4\partial_{ij} U_6 F_{ij}^{(2)}) \\
& + \frac{8}{5}(r^2 x_i \partial_j U_2 - 2x_i \partial_j U_4 - 4x_{ik}\partial_{kj} U_4 - 2r^2 \partial_{ij} U_4 \\
& \quad - 8\partial_{ij} U_6 + 16 x_k \partial_{ijk} U_6) F_{ij}^{(2)} \\
& + \frac{8}{7}(\hat{x}_{ijk}\partial_k U_2 - 6x_{\langle ij}\partial_{k\rangle}\partial_k U_4 + 24 x_{\langle i}\partial_{jk\rangle}\partial_k U_6 - 48\hat{\partial}_{ijk}\partial_k U_8) F_{ij}^{(2)} \\
& + 4(\hat{x}_{ij} U_2 - 4x_{\langle i}\partial_{j\rangle} U_4 + 8\partial_{ij} U_6) F_{ij}^{(2)} \\
& + 4(\hat{x}_{ij}\partial_t^2 U_2 - 4x_{\langle i}\partial_{j\rangle}\partial_t^2 U_4 + 8\partial_{ij}\partial_t^2 U_6) F_{ij} \\
& + 16(\hat{x}_i U_2^j - 2\partial_i U_4^j) F_{ij}^{(1)} \\
& - \frac{40}{21}(\hat{x}_{ijk}\partial_k\partial_t U_2 - 6x_{\langle ij}\partial_{k\rangle}\partial_k\partial_t U_4 + 24 x_{\langle i}\partial_{jk\rangle}\partial_k\partial_t U_6 \\
& \quad - 48\hat{\partial}_{ijk}\partial_k\partial_t U_8) F_{ij}^{(1)} \\
& + \frac{5}{27}(\hat{x}_{ijkl}\partial_{kl} U_2 - 8x_{\langle ijk}\partial_{l\rangle}\partial_{kl} U_4 + 48 x_{\langle ij}\partial_{kl\rangle}\partial_{kl} U_6 \\
& \quad - 192 x_{\langle i}\partial_{jkl\rangle}\partial_{kl} U_8 + 384\hat{\partial}_{ijkl}\partial_{kl} U_{10}) F_{ij}^{(2)} \\
& + 4(\hat{x}_{ij}\partial_t U_2 - 4x_{\langle i}\partial_{j\rangle}\partial_t U_4 + 8\partial_{ij}\partial_t U_6) F_{ij}^{(1)} \\
& + \frac{80}{21}(\hat{x}_{ij}\partial_k U_2^k - 4x_{\langle i}\partial_{j\rangle}\partial_k U_4^k + 8\hat{\partial}_{ij}\partial_k U_6^k) F_{ij}^{(1)} \\
& + \frac{160}{21}(\hat{x}_{ik}\partial_k U_2^j - 4x_{\langle i}\partial_{k\rangle}\partial_k U_4^j + 8\hat{\partial}_{ik}\partial_k U_6^j) F_{ij}^{(1)} \\
& - \frac{64}{21}(\hat{x}_{ik}\partial_j U_2^k - 4x_{\langle i}\partial_{k\rangle}\partial_j U_4^k + 8\hat{\partial}_{ik}\partial_j U_6^k) F_{ij}^{(1)} \\
& + 8(\hat{W}_2^{ij} + x_i\partial_j(\hat{W} + 4U^2)_2 - 2\partial_{ij}(\hat{W} + 4U^2)_4) F_{ij} \Big] + \mathcal{O}\left(\frac{1}{c^{19}}\right),
\end{aligned} \tag{4.25a}$$

$$\delta h^{0i}_{(2)} = \frac{1}{c^{16}}\bigg[-6(x_j\partial_t U_2 - 2\partial_j\partial_t U_4)F_{ij} - 3(x_{jk}\partial_i U_2 - 4x_j\partial_{ik}U_4 + 8\partial_{ijk}U_6)F^{(1)}_{jk} + \frac{8}{3}(\hat{x}_{jk}\partial_k U_2 - 4x_{\langle j}\partial_{k\rangle}\partial_k U_4 + 8\hat{\partial}_{jk}\partial_k U_6)F^{(1)}_{ij} - \frac{8}{3}(\hat{x}_{ij}\partial_k U_2 - 4x_{\langle i}\partial_{j\rangle}\partial_k U_4 + 8\hat{\partial}_{ij}\partial_k U_6)F^{(1)}_{jk} - 8(x_j(\partial_i U^k_2 - \partial_k U^i_2) - 2(\partial_{ij}U^k_4 - \partial_{jk}U^i_4))F_{jk}\bigg] + \mathcal{O}\left(\frac{1}{c^{18}}\right), \tag{4.25b}$$

$$\delta h^{ij}_{(2)} = \frac{1}{c^{15}}[-4(x_k\partial_{(i}U_2 - 2\partial_{k(i}U_4)F_{j)k} + 2\delta_{ij}(x_k\partial_l U_2 - 2\partial_{kl}U_4)F_{kl}] + \mathcal{O}\left(\frac{1}{c^{17}}\right). \tag{4.25c}$$

We must also do the same for the other multipole interactions, but these arise at higher PN order and the iteration is much simpler. We need only consider the mass octupole and current quadrupole moments, for which we define

$$G_{ijk}(t) = \frac{208}{3969}G^3M^2\int_0^{+\infty}\mathrm{d}\tau \ln\tau I^{(10)}_{ijk}(t-\tau), \tag{4.26a}$$

$$H_{ij}(t) = \frac{6848}{4725}G^3M^2\int_0^{+\infty}\mathrm{d}\tau \ln\tau J^{(8)}_{ij}(t-\tau). \tag{4.26b}$$

For the mass octupole moment we obtain

$$\delta h^{00}_{(2)} + \delta h^{ii}_{(2)} = \frac{24}{c^{17}}[x_{ij}\partial_k U_2 - 4x_i\partial_{jk}U_4 + 8\partial_{ijk}U_6]G_{ijk} + \mathcal{O}\left(\frac{1}{c^{19}}\right), \tag{4.27}$$

while the other components are negligible. For the current quadrupole we have

$$\delta h^{00}_{(2)} + \delta h^{ii}_{(2)} = \frac{8}{c^{17}}\bigg[(\hat{x}_{bc}\partial_i\partial_t U_2 - 4x_{\langle b}\partial_{c\rangle}\partial_i\partial_t U_4 + 8\hat{\partial}_{bc}\partial_i\partial_t U_6)\varepsilon_{iab}H_{ac} - \frac{5}{28}(\hat{x}_{jbc}\partial_{ij}U_2 - 6\hat{x}_{\langle jb}\partial_{c\rangle}\partial_{ij}U_4 + 24x_{\langle j}\partial_{bc\rangle}\partial_{ij}U_6 - 48\hat{\partial}_{jbc}\partial_{ij}U_8)\varepsilon_{iab}H^{(1)}_{ac} + 2(x_c\partial_i U^j_2 - 2\partial_{ci}U^j_4)\varepsilon_{ija}H_{ac} - 2(x_b\partial_i U^j_2 - 2\partial_{bi}U^j_4)\varepsilon_{iab}H_{aj}\bigg] + \mathcal{O}\left(\frac{1}{c^{19}}\right), \tag{4.28a}$$

$$\delta h^{0i}_{(2)} = \frac{4}{c^{16}}[-2(x_c\partial_j U_2 - 2\partial_{cj}U_4)\varepsilon_{ija}H_{ac} + (x_b\partial_j U_2 - 2\partial_{bj}U_4)(\varepsilon_{iab}H_{aj} - \varepsilon_{jab}H_{ai})] + \mathcal{O}\left(\frac{1}{c^{18}}\right), \tag{4.28b}$$

while the ij components are negligible.

C. Cubic iteration

At the next-to-next-to-leading 7.5PN order (i.e. 2PN relative) it is evident that there is one further iteration to be performed. However that iteration will concern only the mass quadrupole moment that appears at the leading 5.5PN order. For that moment we have to integrate the cubic equation

$$\Box h^{\mu\nu}_{(3)} = M^{\mu\nu}_{(3)} + N^{\mu\nu}_{(3)}. \tag{4.29}$$

The cubic source term is the sum of two contributions: $M^{\mu\nu}_{(3)}$ which is a direct product of three linear terms $h^{\mu\nu}_{(1)}$ and can be symbolically written as $\sim h_{(1)}\partial h_{(1)}\partial h_{(1)}$, and $N^{\mu\nu}_{(3)}$ which is a product between a linear term $h^{\mu\nu}_{(1)}$ and a quadratic one $h^{\mu\nu}_{(2)}$, symbolically written as $\sim\partial h_{(1)}\partial h_{(2)}$. At cubic order only the dominant contribution in the combination $00+ii$ of the components of the source terms will be needed.

Considering first the $M^{\mu\nu}_{(3)}$ piece we find that the dominant contribution therein is

$$M^{00}_{(3)} + M^{ii}_{(3)} = -\frac{9}{8}h^{00}_{(1)}\partial_i h^{00}_{(1)}\partial_i h^{00}_{(1)}. \tag{4.30}$$

We replace the linear metric $h^{00}_{(1)}$ by its explicit expression made of the sum of Eqs. (4.1a) and (4.7a) (in which only the leading term of order c^{-13} is to be included), and find again that the integration can be explicitly performed thanks to the method of superpotentials. However we need to compute the superpotentials of slightly more complicated potentials with noncompact support. A first series is (with U Newtonian)

$$(U\partial_i U)_2 = \frac{G^2 m_1^2}{2}\partial_i \ln r_1 + \frac{G^2 m_2^2}{2}\partial_i \ln r_2 - G^2 m_1 m_2\left(\frac{\partial g}{\partial y_1^i} + \frac{\partial g}{\partial y_2^i}\right), \tag{4.31a}$$

$$(U\partial_i U)_4 = \frac{G^2 m_1^2}{12}\partial_i\left[r_1^2\left(\ln r_1 - \frac{5}{6}\right)\right] + \frac{G^2 m_2^2}{12}\partial_i\left[r_2^2\left(\ln r_2 - \frac{5}{6}\right)\right] - \frac{G^2 m_1 m_2}{2}\left(\frac{\partial f}{\partial y_1^i} + \frac{\partial f}{\partial y_2^i}\right), \tag{4.31b}$$

where f has been defined by Eq. (4.22). To define another series we introduce the superpotential of $U\Delta U$ namely (at Newtonian order)

$$K = (U\Delta U)_2 = \frac{Gm_1}{r_1}(U)_1 + \frac{Gm_2}{r_2}(U)_2, \tag{4.32}$$

with $(U)_1 = Gm_2/r_{12}$ and $(U)_2 = Gm_1/r_{12}$ being the values of U at the locations of the particles. Notice that in fact (at Newtonian order) K is related to the trace $\hat{W} = \hat{W}_{kk}$ of Eq. (4.2c) by

$$K = \hat{W} + \frac{U^2}{2} + 2U_{kk}. \tag{4.33}$$

Then we can write, with the superpotentials of K computed similarly to Eq. (4.18),

$$(\partial_i U\partial_i U)_2 = -K + \frac{U^2}{2}, \tag{4.34a}$$

$$(\partial_i U\partial_i U)_4 = -K_2 + \frac{G^2 m_1^2}{2}\ln r_1 + \frac{G^2 m_2^2}{2}\ln r_2 + G^2 m_1 m_2 g, \tag{4.34b}$$

$$(\partial_i U\partial_i U)_6 = -K_4 + \frac{G^2 m_1^2}{12} r_1^2\left(\ln r_1 - \frac{5}{6}\right) + \frac{G^2 m_2^2}{12} r_2^2\left(\ln r_2 - \frac{5}{6}\right) + \frac{G^2 m_1 m_2}{2} f. \tag{4.34c}$$

With these results we obtain in fully closed form the solution corresponding to the direct cubic source term (4.30) as

$$\delta h^{00}_{(3)} + \delta h^{ii}_{(3)} = \frac{18}{c^{17}}[-4x_i(U\partial_j U)_2 + 8\partial_i(U\partial_j U)_4 - x_{ij}(\partial_k U\partial_k U)_2 + 4x_i\partial_j(\partial_k U\partial_k U)_4 - 8\partial_{ij}(\partial_k U\partial_k U)_6]F_{ij} + \mathcal{O}\left(\frac{1}{c^{19}}\right), \tag{4.35}$$

with, as we said, the other components $0i$ and ij being negligible at this stage.

Considering next the $N^{\mu\nu}_{(3)}$ piece of the cubic source term (4.29) we find that only the following contributions are needed:

$$N^{00}_{(3)} + N^{ii}_{(3)} = -h^{ij}_{(1)}\partial_{ij}h^{00}_{(2)} - h^{ij}_{(2)}\partial_{ij}h^{00}_{(1)} - 2\partial_i h^{00}_{(1)}\partial_i h^{00}_{(2)}. \tag{4.36}$$

Again the method of superpotentials works for all the terms encountered. The needed superpotentials are some straightforward extensions or variants of the ones in Eqs. (4.31) and (4.34). Let us add that the superpotentials of r_1/r_2 and r_2/r_1 are also needed. These are obtained by appropriate exchanges between the field point $\mathbf{x}$ and the source points $\mathbf{y}_A$ in Eq. (4.22). Posing

$$f_{12} = -\frac{1}{3}\mathbf{r}_1\cdot\mathbf{r}_{12}\left[g - \frac{1}{3}\right] + \frac{1}{6}(r_1 r_2 + r_2 r_{12} - r_1 r_{12}), \tag{4.37a}$$

$$f_{21} = \frac{1}{3}\mathbf{r}_2\cdot\mathbf{r}_{12}\left[g - \frac{1}{3}\right] + \frac{1}{6}(r_1 r_2 + r_1 r_{12} - r_2 r_{12}), \tag{4.37b}$$

we have $\Delta f_{12} = r_1/r_2$ and $\Delta f_{21} = r_2/r_1$. Those solutions appear in the more complicated superpotential

$$(U_2\partial_{ij}U)_2 = -\frac{G^2 m_1^2}{8}\left[\partial_{ij}\left(r_1^2\left(\ln r_1 - \frac{5}{6}\right)\right) - 2\delta_{ij}\ln r_1\right] - \frac{G^2 m_2^2}{8}\left[\partial_{ij}\left(r_2^2\left(\ln r_2 - \frac{5}{6}\right)\right) - 2\delta_{ij}\ln r_2\right] + \frac{G^2 m_1 m_2}{2}\left(\frac{\partial^2 f_{21}}{\partial y_1^{ij}} + \frac{\partial^2 f_{12}}{\partial y_2^{ij}}\right). \tag{4.38a}$$

Finally we encounter a series of superpotentials, with compact support generalizing Eq. (4.32), of the type

$$(\phi\Delta U)_{2k+2} = \frac{1}{(2k)!}[Gm_1(\phi)_1 r_1^{2k-1} + Gm_2(\phi)_2 r_2^{2k-1}], \tag{4.39}$$

where $(\phi)_A$ denotes the value of ϕ at the particle A. We are finally in a position to write down the complete explicit form of the cubic solution of Eq. (4.36) as

$$\delta h^{00}_{(3)} + \delta h^{ii}_{(3)} = \frac{8}{c^{17}}[x_i(7U\partial_j U_2 - U_2\partial_j U) - 14U\partial_{ij}U_4 + 2\partial_i U\partial_j U_4 - 6x_i(U\partial_j U)_2 + 12\partial_i(U\partial_j U)_4 + 2(U_2\partial_{ij}U)_2 - 7x_i(\partial_j U_2\Delta U)_2 + x_i\partial_j(U_2\Delta U)_2 + 14\partial_i(\partial_j U_2\Delta U)_4 - 2\partial_{ij}(U_2\Delta U)_4 + 14(\partial_{ij}U_4\Delta U)_2 - 2\partial_i(\partial_j U_4\Delta U)_2]F_{ij} + \mathcal{O}\left(\frac{1}{c^{19}}\right). \tag{4.40}$$

D. Miscellaneous

A few operations are still in order before obtaining the relevant metric and the result for the redshift factor (1.2). Of course we have to sum up all the results, thereby obtaining the full (iterated and twice-iterated) tail-of-tail contributions in the gothic metric deviation,

$$\delta h^{\mu\nu} = \delta h^{\mu\nu}_{(1)} + \delta h^{\mu\nu}_{(2)} + \delta h^{\mu\nu}_{(3)}, \tag{4.41}$$

where $\delta h^{\mu\nu}_{(1)}$ is itself the sum of Eqs. (4.7) to (4.11), $\delta h^{\mu\nu}_{(2)}$ is the sum of (4.25), (4.27) and (4.28), and $\delta h^{\mu\nu}_{(3)}$ is the sum of (4.35) and (4.40). The corresponding contributions in the usual covariant metric, say $\delta g_{\mu\nu}$, must then be deduced from (4.41). This is a straightforward step and we get, up to the requested PN order,

$$\delta g_{00} = -\frac{1}{2}\left(1 + \frac{h^{00}+h^{ii}}{2}\right)(\delta h^{00} + \delta h^{ii}) - \frac{1}{2}h^{00}\delta h^{00} + h^{0i}\delta h^{0i} + \frac{1}{2}h^{ij}\delta h^{ij} - \frac{15}{16}(h^{00})^2\delta h^{00} + \mathcal{O}\left(\frac{1}{c^{19}}\right), \tag{4.42a}$$

$$\delta g_{0i} = \left(1 + \frac{h^{00}}{2}\right)\delta h^{0i} + \frac{1}{2}h^{0i}\delta h^{00} + \mathcal{O}\left(\frac{1}{c^{18}}\right), \tag{4.42b}$$

$$\delta g_{ij} = -\delta h^{ij} + \frac{1}{2}(-\delta h^{00} + \delta h^{kk})\delta_{ij} - \frac{1}{4}h^{00}\delta h^{00}\delta^{ij} + \mathcal{O}\left(\frac{1}{c^{17}}\right), \tag{4.42c}$$

where $h^{\mu\nu}$ is the 1PN gothic metric (4.1).

Next we have to single out the conservative part of the metric, i.e. neglect the dissipative radiation reaction effects. As in Paper I we assume that the split between conservative and dissipative effects is equivalent to a split between "time-symmetric" and "time-antisymmetric" contributions in the following sense. We decompose each of the tail-of-tail integrals, like for instance F_{ij} defined in Eq. (4.24), into $F_{ij} = F^{\rm cons}_{ij} + F^{\rm diss}_{ij}$ where

$$F^{\rm cons}_{ij}(t) = -\frac{1712}{1050}G^3M^2\int_0^{+\infty} \mathrm{d}\tau \ln\tau [I^{(8)}_{ij}(t-\tau) + I^{(8)}_{ij}(t+\tau)], \tag{4.43a}$$

$$F^{\rm diss}_{ij}(t) = -\frac{1712}{1050}G^3M^2\int_0^{+\infty} \mathrm{d}\tau \ln\tau [I^{(8)}_{ij}(t-\tau) - I^{(8)}_{ij}(t+\tau)], \tag{4.43b}$$

and keep only the conservative part that is time symmetric. This was justified in Paper I by the fact that the equations of motion of compact binaries associated with the conservative part of the metric defined in that way are indeed conservative, i.e. the acceleration is purely radial for circular orbits.

From the equations of motion reduced to circular orbits we obtain the relation between the separation r_{12} between the particles and the orbital frequency Ω. This relation is important when we reduce the expressions to the frame of the center of mass and then to circular orbits. We have checked that the results obtained in Eqs. (5.16)–(5.17) of Paper I are sufficient for the present purpose. However it is important that in all relations (such as the one between orbital separation and frequency) we take into account the lowest order 2PN corrections, appropriate when performing a next-to-next-to-leading computation. For the same reason it is also important, when we replace the complete covariant metric $g_{\mu\nu}$ in the redshift factor defined by Eq. (1.2), to include not only all the high-order tail-of-tail pieces, but also the lower order covariant metric up to 2PN order, because of couplings between the 2PN metric and the various iterated tail-of-tail pieces at next-to-next-to-leading order. We do not reproduce here the 2PN metric at the location of each particle since it is given in full form by Eqs. (7.6) of Ref. [33].

V. DISCUSSION

In this paper, using standard post-Newtonian methods (see e.g. [32]), we have computed next-to-next-to-leading contributions to Detweiler's redshift variable [1] at odd powers in the post-Newtonian expansion, by examining the conservative post-Newtonian dynamics of compact binaries moving on exactly circular orbits. Conservative PN effects at odd powers in the PN expansion necessarily involve nonlocal in time or hereditary (tail) integrals extending over the whole past history of the source [22]. They have been shown to appear first at the 5.5PN order in the redshift factor for circular orbits [15]. In the standard PN approximation they have been proved to originate from the so-called tails of tails associated with the mass quadrupole moment of the source [22].

Here we have extended our previous effort to 2PN order beyond the leading 5.5PN contribution, thus obtaining the 6.5PN and 7.5PN coefficients in the redshift factor (at linear order in the mass ratio), which are perhaps the highest orders ever reached by traditional PN methods. This work involved computing high-order tails of tails associated with higher mass and current multipole moments. For this purpose, we have systematically worked in a preferred gauge for which the computation drastically simplifies, with respect to, say, the harmonic gauge. In addition we have employed a more efficient method to obtain the precise coefficients of tail-of-tail integrals in the near zone of general matter sources. Furthermore, we could perform the nonlinear iteration of tails of tails thanks to an integration method based on the use of hierarchical superpotentials. Our analytical post-Newtonian calculation gives results in full agreement with numerical and analytical self-force calculations [15,18].

The present work is an addition to the body of works [1,3,4,14,22] that have demonstrated the beautiful consistency between analytical post-Newtonian methods, valid for any matter source but limited to the weak-field slow-motion regime of the source, and gravitational self-force methods, which give an accurate description of extreme mass ratio compact binaries even in the relativistic and strong-field regime. The agreement between PN and GSF approaches provides an indirect check that the dimensional regularization procedure invoked in the PN calculation when it is applied to point particle binary sources is in fact equivalent to the very different procedure of subtraction of the singular field which is employed in the GSF approach. Although the dimensional regularization has not been explicitly used in the present paper, this check between very different regularization procedures was a central motivation for our initial works [3,4]. Our recent work [22] together with the present paper confirm that the machinery used in the traditional PN approach to compute nonlinear effects and their associated hereditary-type integrals, like tails, tails of tails and so on, is correct.

In principle, the prospects of extending the present analysis to yet higher PN orders are good. The main challenge would be to control higher nonlinear multipole interactions. In particular, the computation of the coefficient at 8.5PN order would be feasible since we know the mass quadrupole moment to 3PN order. On the other hand, extension to higher order in the mass ratio would be possible only by controlling other multipole couplings such as the double mass quadrupole interaction coupled with mass monopoles.

The success of the comparison performed in this paper has obviously important implications: post-Newtonian calculations of tails of tails at 3PN order beyond the (2.5PN) quadrupole term already play a role [32] in the generation of template waveforms for comparable mass compact binaries (made of neutron stars or black holes) to be analyzed in ground or space based detectors. By contrast, self-force computations are designed with the view to generate waveforms for comparison with the extreme mass ratio inspiral signals expected from future space based detectors.

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APPENDIX A: ALTERNATIVE COMPUTATION OF POLE PART CONTRIBUTIONS

According to our discussion in Sec. II, in the coefficients given by Eq. (2.11), namely

$$C_{k,\ell,m}(B) = \sum_{i=0}^{\ell} \gamma_{k,\ell,i}(B) \int_0^{+\infty} \mathrm{d}y y^i \frac{Q_m(1+y)}{(2+y)^{B-k+2}}, \quad \text{(A1a)}$$

where $\gamma_{k,\ell,i}(B) = \frac{(\ell+i)!}{i!(\ell-i)!} \frac{\Gamma(B-k-\ell+2)}{2^{i+1}\Gamma(B+1)}$

$$\times \frac{\Gamma(B-k+\ell+3)}{\Gamma(B-k+i+3)}, \quad \text{(A1b)}$$

we should compute the pole part when $B \to 0$. Instead of expanding directly Eqs. (A1) when $B \to 0$ (as was done in Paper I), it can be more convenient to expand an equivalent, more explicit, expression of $C_{k,\ell,m}(B)$ that merely differs from the original one by a finite remainder $\mathcal{O}(B^0)$. This suggests an alternative method for computing the poles at $B = 0$, based on the fact that the integral

$$\mathcal{I}_m^\nu \equiv \int_0^{+\infty} \mathrm{d}y y^\nu Q_m(1+y) = 2^\nu \frac{[\Gamma(\nu+1)]^2 \Gamma(m-\nu)}{\Gamma(m+\nu+2)} \quad \text{(A2)}$$

is known by analytic continuation for any $\nu \in \mathbb{C}$, except at isolated poles at integer values of ν (see e.g. [48]). The idea is to reshape the right-hand side of Eq. (A1a) in order to express it in terms of integrals that possess the required form (A2). We proceed in two steps.

(i) We perform the following transformation on the integrand of Eq. (A1a). For $k \geq 2$, we write the denominator in the original integrand as $(2+y)^{k-2}/(2+y)^B$ and expand $(2+y)^{k-2}$ by means of the binomial theorem. This gives ($k \geq 2$)

$$C_{k,\ell,m}(B) = \sum_{i=0}^{\ell} \gamma_{k,\ell,i}(B) \sum_{j=i}^{i+k-2} 2^{i+k-j-2} \binom{k-2}{j-i} \times \int_0^{+\infty} \mathrm{d}y y^j \frac{Q_m(1+y)}{(2+y)^B}, \tag{A3}$$

where $\binom{k-2}{j-i}$ is the usual binomial coefficient. For $k=1$, we replace the factor y^i by the equivalent form $(2+y)\sum_{j=0}^{i-1}(-2)^{i-j-1}y^j + (-2)^i$, which yields

$$C_{1,\ell,m}(B) = \sum_{i=0}^{\ell} \gamma_{1,\ell,i}(B) \sum_{j=0}^{i-1} (-2)^{i-j-1} \times \int_0^{+\infty} \mathrm{d}y y^j \frac{Q_m(1+y)}{(2+y)^B} + \frac{1}{2}\int_0^{+\infty} \mathrm{d}y \frac{Q_m(1+y)}{(2+y)^{B+1}}. \tag{A4}$$

The last integral in this expression corresponds to the contribution to $C_{k,\ell,m}(B)$ produced by the term $(-2)^i$. Its coefficient has been simplified by means of the following identity,

$$\sum_{i=0}^{\ell} (-2)^i \gamma_{1,\ell,i}(B) = \frac{1}{2}, \tag{A5}$$

resulting from the Gauss theorem on hypergeometric functions. Remarkably, the last term in (A4) has no pole at $B=0$ for $m \in \mathbb{N}$, since the integral is well defined in the limit $B \to 0$, and it is therefore irrelevant for our analysis. Thus we shall only need the expression

$$C_{1,\ell,m}(B) = \sum_{i=0}^{\ell} \gamma_{1,\ell,i}(B) \sum_{j=0}^{i-1} (-2)^{i-j-1} \times \int_0^{+\infty} \mathrm{d}y y^j \frac{Q_m(1+y)}{(2+y)^B} + \mathcal{O}(B^0). \tag{A6}$$

With Eqs. (A3) and (A6) in hand, we see that all elementary integrands that may be associated with poles are now of the type $y^j Q_m(1+y)/(2+y)^B$ (with $j \in \mathbb{N}$).

(ii) It is immediately possible to check that the prefactors in Eqs. (A3) and (A6) cannot have more than a simple pole, so that it is sufficient to control the integrals of $y^j Q_m(1+y)/(2+y)^B$ at order $\mathcal{O}(B^0)$, neglecting remainders $\mathcal{O}(B)$. If $j < m$ the integral is convergent when $B=0$ and its value is given by Eq. (A2). The problem is more difficult when $j \geq m+1$. In that case we introduce the asymptotic expansion of $y^j Q_m(1+y)$ when $y \to +\infty$. It is obtained by expanding when $y \to +\infty$ the monomials, say $(1+y)^{-2q-m-1}$ ($q \in \mathbb{N}$), in the hypergeometric series defining $Q_m(1+y)$. After some technical manipulation involving again the Gauss theorem, we get

$$y^j Q_m(1+y) = \sum_{p=0}^{j-m-1} f_{j,m,p} y^p + \mathcal{O}\left(\frac{1}{y}\right), \tag{A7a}$$

$$\text{where } f_{j,m,p} = (-)^m \frac{(-2)^{j-p-1}[(j-p-1)!]^2}{(j-m-p-1)!(j+m-p)!}. \tag{A7b}$$

Concretely, we shall resort to the following lemma, valid in the limit $B \to 0$.

Lemma: $$\int_0^{+\infty} \mathrm{d}y y^j \frac{Q_m(1+y)}{(2+y)^B} = \int_0^{+\infty} \mathrm{d}y y^{j-B} Q_m(1+y) + \sum_{p=0}^{j-m-1} \frac{(-2)^{p+1}}{p+1} f_{j,m,p} + \mathcal{O}(B). \tag{A8}$$

This permits us to relate the remaining integrals in (A3) and (A6) to the simpler integrals that admit the closed-form analytic expression (A2), with $\nu = j - B$. The proof relies on the observation that, in the limit where $B \to 0$,

$$\int_0^{+\infty} \mathrm{d}y \left(\frac{1}{(2+y)^B} - \frac{1}{y^B}\right)\left[y^j Q_m(1+y) - \sum_{p=0}^{j-m-1} f_{j,m,p} y^p\right] = \mathcal{O}(B). \tag{A9}$$

This follows from the fact that the second factor inside the integrand behaves like $\mathcal{O}(1/y)$ when $y \to +\infty$ [see Eq. (A7a)]; so the integral is well defined in a neighborhood of $B=0$ and vanishes at that point. In addition, we can compute explicitly, in the sense of analytic continuation in B and in the limit $B \to 0$,

$$\int_0^{+\infty} \mathrm{d}y y^p \left(\frac{1}{(2+y)^B} - \frac{1}{y^B}\right) = \frac{(-2)^{p+1}}{p+1} + \mathcal{O}(B). \tag{A10}$$

The two facts (A9)–(A10) imply Eq. (A8).

Finally, transforming the integrals that enter Eqs. (A3) and (A6) by means of our lemma (A8), when combined with the expressions (A7b) for the coefficients $f_{j,m,p}$ and (A2) for the integral $\mathcal{I}_m^{j-B}$, we obtain, in the cases $k \geq 2$ and $k=1$ respectively,

$$C_{k,\ell,m}(B) = 2^{k-3}(k-2)!\,\frac{\Gamma(2-k-\ell+B)\Gamma(\ell+3-k+B)}{\Gamma(1+B)} \times \sum_{i=0}^{\ell+k-2} \frac{c_{k,\ell,i}(B)}{i!}\left[\frac{[\Gamma(i+1-B)]^2\Gamma(m-i+B)}{2^B\Gamma(m+i+2-B)} + (-)^{m+i}e_{i,m}\right] + \mathcal{O}(B^0), \tag{A11a}$$

$$C_{1,\ell,m}(B) = \frac{\Gamma(1-\ell+B)\Gamma(\ell+2+B)}{4\Gamma(1+B)} \times \sum_{i=1}^{\ell} d_{\ell,i}(B)\left[\frac{[\Gamma(i-B)]^2\Gamma(m+1-i+B)}{2^B\Gamma(m+i+1-B)} + (-)^{m+i+1}e_{i-1,m}\right] + \mathcal{O}(B^0). \tag{A11b}$$

The coefficients therein read

$$c_{k,\ell,i}(B) = \sum_{j=\max(0,i+2-k)}^{\min(\ell,i)} \binom{i}{j} \frac{(\ell+j)!}{(\ell-j)!(k+j-i-2)!}\frac{1}{\Gamma(j+3-k+B)}, \tag{A12a}$$

$$d_{\ell,i}(B) = \sum_{j=0}^{\ell-i}(-)^j \frac{(\ell+i+j)!}{(\ell-i-j)!(i+j)!}\frac{1}{\Gamma(i+j+2+B)}, \tag{A12b}$$

$$e_{i,m} = \sum_{j=0}^{i-m-1} \frac{[(i-j-1)!]^2}{(j+1)(i-j-m-1)!(m+i-j)!}. \tag{A12c}$$

The Laurent expansion when $B \to 0$ of the explicit sums (A11)–(A12) can be performed rapidly in a straightforward way.

APPENDIX B: SOURCE TERMS FOR THE TAILS OF TAILS

The tail-of-tail terms associated with the various multipole moments I_L or J_L (symbolized by K_L say) obey a wave equation of the type

$$\Box h^{\alpha\beta}_{M\times M\times K_L} = \Lambda^{\alpha\beta}_{M\times M\times K_L}, \tag{B1}$$

where $\Lambda_{M\times M\times K_L}$ is a cubic source term composed of nonlinear interactions between two static mass monopoles M and the time-varying multipole K_L. This source term has been derived in Eqs. (2.14)–(2.16) of Ref. [36] for the tails of tails associated with the mass quadrupole moment I_{ij}, and this result was the basis of the computation of Paper I. In this appendix we provide similar expressions for the sources of the tails of tails associated with the mass moments I_{ijk}, I_{ijkl} and current moments J_{ij}, J_{ijk} that are also required for the present computations. They have been obtained by means of the same algorithm as in the quadrupolar case, using the *xAct* package bundle for *Mathematica* [49]. As in Ref. [36] and Paper I we split the source terms into an instantaneous (local-in-time) part and a hereditary (past-dependent) one, say

$$\Lambda^{\alpha\beta}_{M\times M\times K_L} = \mathcal{I}^{\alpha\beta}_{M\times M\times K_L} + \mathcal{H}^{\alpha\beta}_{M\times M\times K_L}. \tag{B2}$$

(i) Mass quadrupole moment[5]:

$$\mathcal{I}^{00}_{M\times M\times I_{ij}} = M^2 n_{ab} r^{-7}\{-516 I_{ab} - 516 r I^{(1)}_{ab} - 304 r^2 I^{(2)}_{ab} - 76 r^3 I^{(3)}_{ab} + 108 r^4 I^{(4)}_{ab} + 40 r^5 I^{(5)}_{ab}\}, \tag{B3a}$$

[5] We pose $G = c = 1$ in this appendix.

$$\mathcal{I}^{0i}_{M\times M\times I_{ij}} = M^2\hat{n}_{iab}r^{-6}\left\{4I^{(1)}_{ab} + 4rI^{(2)}_{ab} - 16r^2I^{(3)}_{ab} + \frac{4}{3}r^3I^{(4)}_{ab} - \frac{4}{3}r^4I^{(5)}_{ab}\right\} + M^2n_a r^{-6}\left\{-\frac{372}{5}I^{(1)}_{ai} - \frac{372}{5}rI^{(2)}_{ai} - \frac{232}{5}r^2I^{(3)}_{ai} - \frac{84}{5}r^3I^{(4)}_{ai} + \frac{124}{5}r^4I^{(5)}_{ai}\right\}, \tag{B3b}$$

$$\mathcal{I}^{ij}_{M\times M\times I_{ij}} = M^2\hat{n}_{ijab}r^{-5}\left\{-190I^{(2)}_{ab} - 118rI^{(3)}_{ab} - \frac{92}{3}r^2I^{(4)}_{ab} - 2r^3I^{(5)}_{ab}\right\} + M^2\delta_{ij}n_{ab}r^{-5}\left\{\frac{160}{7}I^{(2)}_{ab} + \frac{176}{7}rI^{(3)}_{ab} - \frac{596}{21}r^2I^{(4)}_{ab} - \frac{160}{21}r^3I^{(5)}_{ab}\right\} + M^2\hat{n}_{a(i}r^{-5}\left\{-\frac{312}{7}I^{(2)}_{j)a} - \frac{248}{7}rI^{(3)}_{j)a} + \frac{400}{7}r^2I^{(4)}_{j)a} + \frac{104}{7}r^3I^{(5)}_{j)a}\right\} + M^2r^{-5}\left\{-12I^{(2)}_{ij} - \frac{196}{15}rI^{(3)}_{ij} - \frac{56}{5}r^2I^{(4)}_{ij} - \frac{48}{5}r^3I^{(5)}_{ij}\right\}. \tag{B3c}$$

$$\mathcal{H}^{00}_{M\times M\times I_{ij}} = M^2n_{ab}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{96Q_0I^{(4)}_{ab} + \left[\frac{272}{5}Q_1 + \frac{168}{5}Q_3\right]rI^{(5)}_{ab} + 32Q_2r^2I^{(6)}_{ab}\right\}, \tag{B4a}$$

$$\mathcal{H}^{0i}_{M\times M\times I_{ij}} = M^2\hat{n}_{iab}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{-32Q_1I^{(4)}_{ab} + \left[-\frac{32}{3}Q_0 + \frac{8}{3}Q_2\right]rI^{(5)}_{ab}\right\} + M^2n_a r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{\frac{96}{5}Q_1I^{(4)}_{ai} + \left[\frac{192}{5}Q_0 + \frac{112}{5}Q_2\right]rI^{(5)}_{ai} + 32Q_1r^2I^{(6)}_{ai}\right\}, \tag{B4b}$$

$$\mathcal{H}^{ij}_{M\times M\times I_{ij}} = M^2\hat{n}_{ijab}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{-32Q_2I^{(4)}_{ab} + \left[-\frac{32}{5}Q_1 - \frac{48}{5}Q_3\right]rI^{(5)}_{ab}\right\} + M^2\delta_{ij}n_{ab}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{-\frac{32}{7}Q_2I^{(4)}_{ab} + \left[-\frac{208}{7}Q_1 + \frac{24}{7}Q_3\right]rI^{(5)}_{ab}\right\} + M^2\hat{n}_{a(i}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{\frac{96}{7}Q_2I^{(4)}_{j)a} + \left[\frac{2112}{35}Q_1 - \frac{192}{35}Q_3\right]rI^{(5)}_{j)a}\right\} + M^2r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{\frac{32}{5}Q_2I^{(4)}_{ij} + \left[\frac{512}{25}Q_1 - \frac{32}{25}Q_3\right]rI^{(5)}_{ij} + 32Q_0r^2I^{(6)}_{ij}\right\}. \tag{B4c}$$

(ii) Mass octupole:

$$\mathcal{I}^{00}_{M\times M\times I_{ijk}} = M^2\hat{n}_{abc}r^{-8}\left\{-1140I_{abc} - 1140rI^{(1)}_{abc} - 616r^2I^{(2)}_{abc} - 236r^3I^{(3)}_{abc} + \frac{76}{3}r^4I^{(4)}_{abc} + \frac{484}{9}r^5I^{(5)}_{abc} + \frac{112}{9}r^6I^{(6)}_{abc}\right\}, \tag{B5a}$$

$$\mathcal{I}^{0i}_{M\times M\times I_{ijk}} = M^2\hat{n}_{iabc}r^{-7}\left\{6I^{(1)}_{abc} + 6rI^{(2)}_{abc} - \frac{37}{3}r^2I^{(3)}_{abc} - \frac{43}{3}r^3I^{(4)}_{abc} - \frac{16}{9}r^4I^{(5)}_{abc} - \frac{1}{3}r^5I^{(6)}_{abc}\right\}$$
$$+ M^2\hat{n}_{ab}r^{-7}\left\{-\frac{892}{7}I^{(1)}_{abi} - \frac{892}{7}rI^{(2)}_{abi} - \frac{492}{7}r^2I^{(3)}_{abi} - \frac{584}{21}r^3I^{(4)}_{abi} + \frac{568}{63}r^4I^{(5)}_{abi} + \frac{572}{63}r^5I^{(6)}_{abi}\right\}, \tag{B5b}$$

$$\mathcal{I}^{ij}_{M\times M\times I_{ijk}} = M^2\hat{n}_{ijabc}r^{-6}\left\{-186I^{(2)}_{abc} - 186rI^{(3)}_{abc} - 68r^2I^{(4)}_{abc} - \frac{34}{3}r^3I^{(5)}_{abc} - \frac{8}{15}r^4I^{(6)}_{abc}\right\}$$
$$+ M^2\delta_{ij}\hat{n}_{abc}r^{-6}\left\{24I^{(2)}_{abc} + 24rI^{(3)}_{abc} - \frac{38}{9}r^2I^{(4)}_{abc} - \frac{346}{27}r^3I^{(5)}_{abc} - \frac{46}{27}r^4I^{(6)}_{abc}\right\}$$
$$+ M^2\hat{n}_{ab(i}r^{-6}\left\{-\frac{140}{3}I^{(2)}_{j)ab} - \frac{140}{3}rI^{(3)}_{j)ab} + \frac{38}{3}r^2I^{(4)}_{j)ab} + \frac{230}{9}r^3I^{(5)}_{j)ab} + \frac{10}{3}r^4I^{(6)}_{j)ab}\right\}$$
$$+ M^2n_ar^{-6}\left\{-\frac{356}{21}I^{(2)}_{aij} - \frac{356}{21}rI^{(3)}_{aij} - \frac{1028}{105}r^2I^{(4)}_{aij} - \frac{296}{63}r^3I^{(5)}_{aij} + \frac{24}{7}r^4I^{(6)}_{aij}\right\}. \tag{B5c}$$

$$\mathcal{H}^{00}_{M\times M\times I_{ijk}} = M^2\hat{n}_{abc}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{32I^{(5)}_{abc}Q_1 + \frac{8}{3}[7Q_2 + 4Q_4]rI^{(6)}_{abc} + \frac{32}{3}Q_3r^2I^{(7)}_{abc}\right\}, \tag{B6a}$$

$$\mathcal{H}^{0i}_{M\times M\times I_{ijk}} = M^2\hat{n}_{iabc}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{-\frac{64}{3}Q_2I^{(5)}_{abc} - \frac{8}{15}[8Q_1 - 3Q_3]rI^{(6)}_{abc}\right\}$$
$$+ M^2\hat{n}_{ab}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{\frac{256}{21}Q_2I^{(5)}_{abi} + \frac{8}{105}[172Q_1 + 93Q_3]rI^{(6)}_{abi} + \frac{32}{3}Q_2r^2I^{(7)}_{abi}\right\}, \tag{B6b}$$

$$\mathcal{H}^{ij}_{M\times M\times I_{ijk}} = M^2\hat{n}_{ijabc}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{-16Q_3I^{(5)}_{abc} - \frac{16}{21}[3Q_2 + 4Q_4]rI^{(6)}_{abc}\right\}$$
$$+ M^2\delta_{ij}\hat{n}_{abc}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{-\frac{16}{9}Q_3I^{(5)}_{abc} - \frac{8}{27}[33Q_2 - 4Q_4]rI^{(6)}_{abc}\right\}$$
$$+ M^2\hat{n}_{ab(i}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{\frac{16}{3}Q_3I^{(5)}_{j)ab} + \frac{32}{63}[39Q_2 - 4Q_4]rI^{(6)}_{j)ab}\right\}$$
$$+ M^2n_ar^{-3}\int_1^{+\infty}\mathrm{d}x\left\{\frac{128}{35}Q_3I^{(5)}_{aij} + \frac{32}{735}[187Q_2 - 12Q_4]rI^{(6)}_{aij} + \frac{32}{3}Q_1r^2I^{(7)}_{aij}\right\}. \tag{B6c}$$

(iii) Mass hexadecapole:

$$\mathcal{I}^{00}_{M\times M\times I_{ijkl}} = M^2\hat{n}_{abcd}r^{-9}\left\{-2520I_{abcd} - 2520rI^{(1)}_{abcd} - 1318r^2I^{(2)}_{abcd} - 478r^3I^{(3)}_{abcd} - \frac{1015}{18}r^4I^{(4)}_{abcd} + \frac{845}{18}r^5I^{(5)}_{abcd} + \frac{133}{6}r^6I^{(6)}_{abcd} + \frac{29}{9}r^7I^{(7)}_{abcd}\right\}, \tag{B7a}$$

$$\mathcal{I}^{0i}_{M\times M\times I_{ijkl}} = M^2\hat{n}_{iabcd}r^{-8}\left\{9I^{(1)}_{abcd} + 9rI^{(2)}_{abcd} - \frac{29}{2}r^2I^{(3)}_{abcd} - \frac{35}{2}r^3I^{(4)}_{abcd} - \frac{68}{9}r^4I^{(5)}_{abcd} - \frac{77}{90}r^5I^{(6)}_{abcd} - \frac{1}{15}r^6I^{(7)}_{abcd}\right\} + M^2\hat{n}_{abc}r^{-8}\left\{-230I^{(1)}_{abci} - 230rI^{(2)}_{abci} - \frac{1093}{9}r^2I^{(3)}_{abci} - \frac{403}{9}r^3I^{(4)}_{abci} - \frac{119}{81}r^4I^{(5)}_{abci} + \frac{646}{81}r^5I^{(6)}_{abci} + \frac{70}{27}r^6I^{(7)}_{abci}\right\}, \tag{B7b}$$

$$\mathcal{I}^{ij}_{M\times M\times I_{ijkl}} = M^2\hat{n}_{ijabcd}r^{-7}\left\{-293I^{(2)}_{abcd} - 293rI^{(3)}_{abcd} - \frac{379}{3}r^2I^{(4)}_{abcd} - \frac{86}{3}r^3I^{(5)}_{abcd} - \frac{146}{45}r^4I^{(6)}_{abcd} - \frac{1}{9}r^5I^{(7)}_{abcd}\right\} + M^2\delta_{ij}\hat{n}_{abcd}r^{-7}\left\{\frac{345}{11}I^{(2)}_{abcd} + \frac{345}{11}rI^{(3)}_{abcd} + \frac{355}{198}r^2I^{(4)}_{abcd} - \frac{1715}{198}r^3I^{(5)}_{abcd} - \frac{4087}{990}r^4I^{(6)}_{abcd} - \frac{53}{165}r^5I^{(7)}_{abcd}\right\} + M^2\hat{n}_{abc(i}r^{-7}\left\{-\frac{672}{11}I^{(2)}_{j)abc} - \frac{672}{11}rI^{(3)}_{j)abc} - \frac{208}{99}r^2I^{(4)}_{j)abc} + \frac{1808}{99}r^3I^{(5)}_{j)abc} + \frac{452}{55}r^4I^{(6)}_{j)abc} + \frac{104}{165}r^5I^{(7)}_{j)abc}\right\} + M^2\hat{n}_{ab}r^{-7}\left\{-\frac{74}{3}I^{(2)}_{abij} - \frac{74}{3}rI^{(3)}_{abij} - \frac{835}{63}r^2I^{(4)}_{abij} - \frac{317}{63}r^3I^{(5)}_{abij} + \frac{110}{189}r^4I^{(6)}_{abij} + \frac{314}{189}r^5I^{(7)}_{abij}\right\}. \tag{B7c}$$

$$\mathcal{H}^{00}_{M\times M\times I_{ijkl}} = M^2\hat{n}_{abcd}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{8Q_2I^{(6)}_{abcd} + \frac{2}{27}[64Q_3 + 35Q_5]rI^{(7)}_{abcd} + \frac{8}{3}Q_4r^2I^{(8)}_{abcd}\right\}, \tag{B8a}$$

$$\mathcal{H}^{0i}_{M\times M\times I_{ijkl}} = M^2\hat{n}_{iabcd}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{-8Q_3I^{(6)}_{abcd} - \frac{2}{21}[12Q_2 - 5Q_4]rI^{(7)}_{abcd}\right\} + M^2\hat{n}_{abc}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{\frac{40}{9}Q_3I^{(6)}_{abci} + \frac{8}{189}[78Q_2 + 41Q_4]rI^{(7)}_{abci} + \frac{8}{3}Q_3r^2I^{(8)}_{abci}\right\}, \tag{B8b}$$

$$\mathcal{H}^{ij}_{M\times M\times I_{ijkl}} = M^2\hat{n}_{ijabcd}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{-\frac{16}{3}Q_4 I^{(6)}_{abcd} - \frac{4}{27}[4Q_3+5Q_5]rI^{(7)}_{abcd}\right\}$$
$$+ M^2\delta_{ij}\hat{n}_{abcd}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{-\frac{16}{33}Q_4 I^{(6)}_{abcd} - \frac{10}{33}[8Q_3-Q_5]rI^{(7)}_{abcd}\right\}$$
$$+ M^2\hat{n}_{abc(i}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{\frac{16}{11}Q_4 I^{(6)}_{j)abc} + \frac{16}{297}[91Q_3-10Q_5]rI^{(7)}_{j)abc}\right\}$$
$$+ M^2\hat{n}_{ab}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{\frac{80}{63}Q_4 I^{(6)}_{abij} + \frac{16}{567}[77Q_3-5Q_5]rI^{(7)}_{abij} + \frac{8}{3}Q_2 r^2 I^{(8)}_{abij}\right\}. \tag{B8c}$$

(iv) Current quadrupole:

$$\mathcal{I}^{00}_{M\times M\times J_{ij}} = 0, \tag{B9a}$$

$$\mathcal{I}^{0i}_{M\times M\times J_{ij}} = M^2\varepsilon_{iab}\hat{n}_{ac}r^{-7}\left\{88J_{bc} + 88rJ^{(1)}_{bc} + 80r^2J^{(2)}_{bc} + \frac{152}{3}r^3J^{(3)}_{bc} - \frac{368}{9}r^4J^{(4)}_{bc} - \frac{208}{9}r^5J^{(5)}_{bc}\right\}, \tag{B9b}$$

$$\mathcal{I}^{ij}_{M\times M\times J_{ij}} = M^2\varepsilon_{ab(i}\hat{n}_{j)ac}r^{-6}\left\{\frac{64}{3}J^{(1)}_{bc} + \frac{64}{3}rJ^{(2)}_{bc} - 64r^2J^{(3)}_{bc} - \frac{608}{9}r^3J^{(4)}_{bc} - \frac{64}{9}r^4J^{(5)}_{bc}\right\}$$
$$+ M^2\varepsilon_{ab(i}n_{\underline{a}}r^{-6}\left\{\frac{304}{15}J^{(1)}_{j)b} + \frac{304}{15}rJ^{(2)}_{j)b} + \frac{368}{15}r^2J^{(3)}_{j)b} + \frac{832}{45}r^3J^{(4)}_{j)b} - \frac{96}{5}r^4J^{(5)}_{j)b}\right\}. \tag{B9c}$$

$$\mathcal{H}^{00}_{M\times M\times J_{ij}} = 0, \tag{B10a}$$

$$\mathcal{H}^{0i}_{M\times M\times J_{ij}} = M^2\varepsilon_{iab}\hat{n}_{ac}r^{-3}\int_1^{+\infty}\mathrm{d}x\left\{\frac{64}{3}Q_2J^{(4)}_{bc} - \frac{64}{15}[7Q_1+3Q_3]rJ^{(5)}_{bc} - \frac{64}{3}Q_2r^2J^{(6)}_{bc}\right\}, \tag{B10b}$$

$$\mathcal{H}^{ij}_{M\times M\times J_{ij}} = M^2\varepsilon_{ab(i}\hat{n}_{j)ac}r^{-2}\int_1^{+\infty}\mathrm{d}x\left\{-\frac{128}{3}Q_2J^{(5)}_{bc}\right\}$$
$$+ M^2\varepsilon_{ab(i}n_{\underline{a}}r^{-2}\int_1^{+\infty}\mathrm{d}x\left\{-\frac{128}{15}Q_2J^{(5)}_{j)b} - \frac{128}{3}Q_1rJ^{(6)}_{j)b}\right\}. \tag{B10c}$$

(v) Current octupole:

$$\mathcal{I}^{00}_{M\times M\times J_{ijk}} = 0, \tag{B11a}$$

$$\mathcal{I}^{0i}_{M\times M\times J_{ijk}} = M^2\varepsilon_{iab}\hat{n}_{acd}r^{-8}\left\{270J_{bcd} + 270rJ^{(1)}_{bcd} + 188r^2J^{(2)}_{bcd} + 98r^3J^{(3)}_{bcd} - 6r^4J^{(4)}_{bcd} - \frac{100}{3}r^5J^{(5)}_{bcd} - 9r^6J^{(6)}_{bcd}\right\}, \tag{B11b}$$

$$\mathcal{I}^{ij}_{M\times M\times J_{ijk}} = M^2\varepsilon_{ab(i}\hat{n}_{j)acd}r^{-7}\Big\{54J^{(1)}_{bcd} + 54rJ^{(2)}_{bcd} - 60r^2J^{(3)}_{bcd} - 78r^3J^{(4)}_{bcd} - 30r^4J^{(5)}_{bcd} - 2r^5J^{(6)}_{bcd}\Big\} + M^2\varepsilon_{ab(i}\hat{n}_{\underline{a}\,\underline{c}}r^{-7}\Big\{\frac{360}{7}J^{(1)}_{j)bc} + \frac{360}{7}rJ^{(2)}_{j)bc} + \frac{286}{7}r^2J^{(3)}_{j)bc} + \frac{166}{7}r^3J^{(4)}_{j)bc} - \frac{32}{7}r^4J^{(5)}_{j)bc} - \frac{236}{21}r^5J^{(6)}_{j)bc}\Big\}. \tag{B11c}$$

$$\mathcal{H}^{00}_{M\times M\times J_{ijk}} = 0, \tag{B12a}$$

$$\mathcal{H}^{0i}_{M\times M\times J_{ijk}} = M^2\varepsilon_{iab}\hat{n}_{acd}r^{-3}\int_1^{+\infty}\mathrm{d}x\Big\{8Q_3J^{(5)}_{bcd} - \frac{16}{7}[5Q_2+2Q_4]rJ^{(6)}_{bcd} - 8Q_3r^2J^{(7)}_{bcd}\Big\}, \tag{B12b}$$

$$\mathcal{H}^{ij}_{M\times M\times J_{ijk}} = M^2\varepsilon_{ab(i}\hat{n}_{j)acd}r^{-2}\int_1^{+\infty}\mathrm{d}x\{-16Q_3J^{(6)}_{bcd}\} + M^2\varepsilon_{ab(i}\hat{n}_{\underline{a}\,\underline{c}}r^{-2}\int_1^{+\infty}\mathrm{d}x\Big\{-\frac{32}{7}Q_3J^{(6)}_{j)bc} - 16Q_2rJ^{(7)}_{j)bc}\Big\}. \tag{B12c}$$

In the hereditary terms the kernels of the integrals are made of Legendre functions of the second kind $Q_m(x)$, see Eq. (2.8), multiplied by time derivatives of the multipole moments evaluated at time $t - rx$.

In Appendix A of Paper I, it was proved that certain specific terms, namely those coming from the second term in Eq. (2.4), do not contribute at half-integral PN orders. It is easy to verify that the proof there applies in the more general case investigated here, where we have additional multipole components besides the mass quadrupole. Indeed we observe that for all the hereditary terms in Eqs. (B3)–(B12) the combination $k + m + \ell$ is always an odd integer, where k represents the power of $1/r$ in the term in question, m is the order of the Legendre function therein and ℓ is the multipolarity of the term. Thus the proof of Appendix A in Paper I can be repeated exactly as it is. This shows that the PN order of the second term in Eq. (2.4), for all these multipole interactions, is necessarily integral and can be ignored in the present computation.

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