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## Field equations and cosmology for a class of nonlocal metric models of MOND

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We consider a class of nonlocal, pure-metric modified gravity models which were developed to reproduce the Tully-Fisher relation without dark matter and without changing the amount of weak lensing predicted by general relativity. Previous work gave only the weak field limiting form of the field equations specialized to a static and spherically symmetric geometry. Here we derive the full field equations and specialize them to a homogeneous, isotropic and spatially flat geometry. We also discuss the problem of fitting the free function to reproduce the expansion history. Results are derived for models in which the MOND acceleration  $a_0 \approx 1.2 \times 10^{-10} \text{ m.s}^{-2}$  is a fundamental constant and for the more phenomenologically interesting case in which the MOND acceleration changes with the cosmological expansion rate.

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### I. INTRODUCTION

Despite the successes of the standard model of cosmology based on general relativity, many feel unsatisfied that the only currently available evidences for dark matter and dark energy are indirect, and it is certainly worth pursuing other approaches. One of them is the MOND paradigm (modified Newtonian dynamics), as proposed by Milgrom [1], which led to many successful explanations of various observations as well as to predictions which were confirmed [2]. In particular, it explains the Tully-Fisher relation [3], which states that the observed limiting rotation velocity of galaxies,  $v_\infty$ , scales as the fourth root of the baryonic mass of the galaxy (see [4] for a recent confirmation of this relation). As it was first formulated in a nonrelativistic way, Milgrom's proposal stipulates that a test particle at a distance  $r$  from a mass  $M$  will experience a gravitational acceleration given by the Newtonian expression  $a_N = GM/r^2$  as long as  $a_N$  is (much) larger than a critical acceleration  $a_0$ , while the same particle will undergo the MOND acceleration  $a_{\text{MOND}} = \sqrt{a_N a_0} = \sqrt{GM a_0}/r$  when  $a_N$  is smaller than  $a_0$ . A constant value for  $a_0$  of about  $1.2 \times 10^{-10} \text{ m.s}^{-2}$  leads to good fits of galaxy rotation curves using reasonable mass-to-luminosity ratios [5], without the need for nonbaryonic dark matter [6]. As noticed by many authors, this numerical value for  $a_0$  is very close to  $cH_0/2\pi$ , where  $H_0$  is the current value of the Hubble parameter. This gives some support to the idea that the MOND parameter  $a_0$  actually varies with time over the

cosmological history of the Universe [7], and we will discuss some aspects of this possibility in this paper (together with the constant  $a_0$  case). Such a time variation of  $a_0$  could also lead to some specific observational signatures [7,8].

Despite its successes, the current MOND framework also suffers from various problems both at the level of observation fitting and theoretical construction. It is indeed well known that some amount of dark matter is needed (or possibly a deviation from the original MOND formulation without dark matter) in order for MOND to fit velocity dispersion in galaxy clusters. The bullet cluster [9] is also often presented as a serious puzzle for MOND (see however [10–12]). On the theory side, the great challenge has been to construct a relativistic extension of MOND that reproduces, without dark matter, both the observed cosmology and the observed amount of weak lensing. Some attempts along this line include TeVeS theories [13–20] and other models with scalar and vector fields [21,22], Milgrom's bi-metric model [23], and nonlocal, metric-based models [24–27]. It is fair to say that these attempts, however very interesting, need to be further explored and consolidated.

In our previous work [28], we introduced yet a new relativistic formulation of MOND which is at the root of the present work. In this formulation, the only dynamical degrees of freedom are those of a metric. Hence, we called such a theory a “pure-metric” theory, in contrast to the TeVeS model or bi-metric theory, which both contain degrees of freedom that are added explicitly to those of the metric. Note that such a distinction cannot always be considered as very deep; e.g., it is well known that  $f(R)$  theories (which would qualify as a pure-metric theory using

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our terminology) can also be formulated as scalar-tensor theories, i.e., as theories with a metric and an extra scalar in the gravitational sector. We will in turn use here sometimes scalars to describe our model. However our scalars will not have a proper dynamics as we will explain later. The advantages of a pure metric-based theory are that it allows a clear way to build the matter coupling in agreement with the equivalence principle as well as a simple comparison with general relativity. As argued in particular in [28,29], it can however be shown that a pure metric based theory of MOND has to be nonlocal,<sup>1</sup> and this is the case in the theory under consideration. The point of this paper is to further develop a class of generally coordinate invariant, nonlocal metric realizations of MOND which have been proposed in [28].

Nonlocal metric extensions of gravity (irrespective of MOND) have been much studied [30] because they offer a richer phenomenology than  $f(R)$  models [31,32], which are the only local, invariant, metric-based and kinetically stable extensions of general relativity [33,34]. We do not believe fundamental theory is nonlocal, but rather that nonlocal extensions of general relativity derive from quantum infrared corrections to the effective field equations that became nonperturbatively strong during an extended phase of primordial inflation [35]. Put simply, we believe that MOND derives from the gravitational vacuum polarization of the vast ensemble of infrared gravitons created during primordial inflation. Although our class of models is, at this stage, purely phenomenological, our suspicion about its probable origin helps to justify two features that would otherwise be inexplicable:

- (i) Our models possess an initial time  $t_i$ , and
- (ii) Our models predict significant deviations from general relativity on large scales but not on small scales.

Our previous work [28] began by deriving phenomenological equations which any metric-based theory of MOND must obey for static, spherically symmetric geometries of the form

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = -[1 + b(r)]dt^2 + [1 + a(r)]dr^2 + r^2 d\Omega^2. \quad (1)$$

If the energy density  $\rho(r)$  is such that the system is everywhere in the MOND regime (as would be the case, for example, in a low surface brightness galaxy) then the Tully-Fisher relation implies [28]

$$\frac{1}{2a_0 r^2} \frac{d}{dr} ([rb'(r)]^2) = 8\pi G \rho(r). \quad (2)$$

<sup>1</sup>As explained in Sec. III of Ref. [28], no local term (built from the metric, its curvature tensor or even its derivatives) can bear simultaneously the right power of the gravitational field and the right number of derivatives to yield the MOND law.

(This phenomenological realization of the Tully-Fisher relation is a powerful motivation for nonlocality because it contains a net three derivatives, as opposed to the nonzero even number of derivatives that must follow from the variation of a local curvature invariant.) We fixed the other potential  $a(r)$  by requiring that weak lensing agrees exactly with what general relativity predicts assuming the potential  $b(r)$  is known,

$$rb'(r) - a(r) = 0. \quad (3)$$

Note that any power of (3) would work as well because the right-hand side of the equation vanishes. If one allows only an approximate agreement with general relativity as far as lensing is concerned, as e.g. is allowed by cosmological data using lensing, a coefficient of order one can just be inserted in front of  $a(r)$  in the equation above.

We assumed a Lagrangian consisting of general relativity and normal matter, plus a MOND correction term  $\Delta\mathcal{L}$ ,

$$\mathcal{L} = \frac{R\sqrt{-g}}{16\pi G} + \Delta\mathcal{L} + \mathcal{L}_{\text{matter}}. \quad (4)$$

Because the matter Lagrangian is unchanged from general relativity (except for the absence of dark matter) our field equations take the form

$$G_{\mu\nu} + \Delta G_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (5)$$

where  $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$  is the usual Einstein tensor and  $T_{\mu\nu}$  is the usual stress-energy tensor. (We employ a metric with mostly plus signature with Riemann tensor  $R^\rho{}_{\sigma\mu\nu} = +\partial_\mu\Gamma^\rho{}_{\nu\sigma} - \dots$  and Ricci tensor  $R_{\mu\nu} \equiv R^\rho{}_{\mu\rho\nu}$ .) The MOND correction  $\Delta G_{\mu\nu}$  to the Einstein tensor comes from varying the action deriving from  $\Delta\mathcal{L}$ , namely, the spacetime integral  $\Delta S$  of  $\Delta\mathcal{L}$ . We get

$$\Delta G_{\mu\nu}(x) \equiv \frac{16\pi G}{\sqrt{-g}} \frac{\delta\Delta S[g]}{\delta g^{\mu\nu}(x)}. \quad (6)$$

We constructed the MOND correction  $\Delta\mathcal{L}$  such that, in the static, spherically symmetric and ultraweak field regime, the  $\mu = \nu = 0$  equation reduces to (2) and the  $\mu = \nu = r$  equation is proportional to (3).

We found that it sufficed to employ a single nonlocal scalar,

$$Y[g] \equiv g^{\mu\nu} \partial_\mu \frac{2}{\square} [u^\alpha u^\beta R_{\alpha\beta}] \partial_\nu \frac{2}{\square} [u^\rho u^\sigma R_{\rho\sigma}]. \quad (7)$$

Here and henceforth  $\square \equiv \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu)$  is the covariant scalar d'Alembertian, and its inverse is defined with retarded boundary conditions on the initial value surface. The timelike four-velocity field  $u^\mu[g]$  is the normalized gradient of some nonlocal scalar functional of the metric

$\chi[g]$ , such as the invariant volume of the past light cone, which grows in the timelike direction,

$$u^\mu[g] \equiv \frac{-g^{\mu\nu} \partial_\nu \chi[g]}{\sqrt{-g^{\alpha\beta} \partial_\alpha \chi[g] \partial_\beta \chi[g]}}. \quad (8)$$

In the static, spherically symmetric and ultraweak field limit,  $Y[g]$  reduces to just  $[b'(r)]^2$  and we could reproduce (2), without disturbing the general relativistic relation (3), with a MOND addition of the form,

$$\Delta \mathcal{L}_y = \frac{a_0^2}{16\pi G} \left\{ \frac{1}{2} \left( \frac{Y}{a_0^2} \right) - \frac{1}{6} \left( \frac{Y}{a_0^2} \right)^{\frac{3}{2}} + \dots \right\} \sqrt{-g}. \quad (9)$$

In this expression, the first term is needed to cancel an equivalent contribution coming from the Einstein-Hilbert action, while the second term is responsible for the MOND force law that one can read from (2).

We found that it was permissible, but not necessary, to involve a second nonlocal scalar,

$$X[g] \equiv g^{\mu\nu} \partial_\mu \frac{1}{\square} \left[ u^\alpha u^\beta R_{\alpha\beta} - \frac{1}{2} R \right] \partial_\nu \frac{1}{\square} \left[ u^\rho u^\sigma R_{\rho\sigma} - \frac{1}{2} R \right]. \quad (10)$$

In the static, spherically symmetric and ultraweak field limit  $X[g]$  reduces to  $[b'(r) - a(r)/r]^2$ . In this limit, the contribution of the Einstein-Hilbert action imposing Eq. (3) (from the  $g_{rr}$  equation) can thus be canceled by adding a term linear in  $X[g]$ , with a suitable coefficient. To this term, one can then add a next-order correction in the ultraweak field expansion, such as

$$\Delta \mathcal{L}_x = \frac{a_0^2}{16\pi G} \left\{ -\frac{1}{2} \left( \frac{X}{a_0^2} \right) + \frac{1}{6} \left( \frac{X}{a_0^2} \right)^{\frac{3}{2}} + \dots \right\} \sqrt{-g}. \quad (11)$$

Any successful implementation of MOND must involve the addition of (9), but the decision of whether or not to additionally include (11) is optional because (3) holds both in general relativity and in the MOND regime (see [28] for more details). Avoiding deviations from existing tests of general relativity requires that the higher order terms give suppression for large values of  $Y[g]/a_0^2$ .

The purpose of this paper is to extend our past results by deriving the MOND corrections to the field equations for a general metric and then specialize them to the homogeneous and isotropic geometry appropriate to cosmology. To keep the analysis simple we define the scalar  $\chi[g]$ , whose normalized gradient gives the timelike four-velocity (8), using the same inverse of the scalar d'Alembertian which appears in both  $Y[g]$  and  $X[g]$ ,

$$\chi[g] \equiv -\frac{1}{\square} 1. \quad (12)$$

We also consider the important changes which occur when one alters the MOND acceleration  $a_0$  from a fundamental constant to a dynamical quantity which varies with the cosmological expansion rate. As mentioned above, many authors have drawn attention to the numerical coincidence  $a_0 \approx cH_0/2\pi$  between the MOND acceleration and the current value of the Hubble parameter [7].<sup>2</sup> With a timelike four-velocity field such as  $u^\mu[g]$ , whose divergence  $D_\mu u^\mu = 3H$  in a cosmological background, it is easy to make this relation dynamical by the replacement

$$a_0 \longrightarrow \alpha[g] \equiv \frac{D_\mu u^\mu}{6\pi}. \quad (13)$$

In this way the extra MOND force, which is necessary if there is no dark matter, can become effective even at early times during which the condition  $|Y/a_0^2| \gg 1$  would otherwise have suppressed MOND effects.

This paper contains five sections of which this introduction is the first. In Sec. II we consider the simplest class of models in which  $\Delta \mathcal{L}$  depends only on the invariant  $Y[g]$ , with the MOND acceleration  $a_0$  a fundamental constant. We derive the correction  $\Delta G_{\mu\nu}$  to the field equations for a general metric and then specialize this to cosmology. Section III carries out the same exercise for MOND additions which also depend on the invariant  $X[g]$ , again with constant  $a_0$ . In Sec. IV we derive the changes which occur when the MOND acceleration is made dynamical through the replacement (13). Section V gives our conclusions.

## II. MODELS BASED ON $Y$ WITH CONSTANT $a_0$

The task of this section is to analyze the minimal class of models,

$$\Delta \mathcal{L}_y = \frac{1}{16\pi G} \times a_0^2 f_y \left( \frac{Y[g]}{a_0^2} \right) \sqrt{-g}, \quad (14)$$

where  $Y[g]$  is the nonlocal invariant defined by expressions (7), (8) and (12), and  $a_0$  is strictly constant. We first express the nonlocal model (14) in a local form involving the metric and four auxiliary scalars. We next vary with respect to  $g^{\mu\nu}$  to derive the MOND addition to the Einstein tensor (6) for a general metric, then specialize to the homogeneous, isotropic and spatially flat geometry appropriate to cosmology. The section closes with a discussion of how the function  $f_y(Z)$  can be chosen for  $Z < 0$  (MOND phenomenology only fixes  $f_y(Z)$  for  $Z > 0$ ) to support an arbitrary expansion history.

<sup>2</sup>As underlined in the first Ref. [7], a slower than Hubble rate evolution of  $a_0$  is allowed by present data on rotation curves of galaxies at high redshifts. We will mention at the end of Sec. IV some ways of reducing this evolution rate as compared to (13).

### A. General field equations

We can derive causal and conserved field equations from the nonlocal form (14) using the ‘‘partial integration trick’’ of earlier studies [36–38]. However, it is simpler to localize  $\Delta\mathcal{L}_y$  using scalar auxiliary fields after the procedure of Nojiri and Odintsov [39]. Our model (14) requires scalars  $\phi$  and  $\chi$  to stand for the two nonlocal expressions in the original Lagrangian,

$$\phi \longrightarrow \frac{2}{\square} u^\alpha u^\beta R_{\alpha\beta}, \quad \chi \longrightarrow -\frac{1}{\square} 1, \quad (15)$$

and Lagrange multiplier fields  $\xi$  and  $\psi$  to enforce these relations. We shall abuse the notation slightly by employing the same symbol for the local Lagrangian and its nonlocal ancestor (14),

$$\Delta\mathcal{L}_y = \frac{1}{16\pi G} \left\{ a_0^2 f_y \left( \frac{g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi}{a_0^2} \right) - [\partial_\mu \xi \partial_\nu \phi g^{\mu\nu} + 2\xi R_{\mu\nu} u^\mu u^\nu] - [\partial_\mu \psi \partial_\nu \chi g^{\mu\nu} - \psi] \right\} \sqrt{-g}. \quad (16)$$

The four-velocity field in this version of the model is still the normalized gradient (8) of  $\chi$ , but the scalar  $\chi$  is an independent variable.

It is straightforward to compute the MOND correction to the Einstein tensor,

$$\begin{aligned} \frac{16\pi G \delta\Delta S_y}{\sqrt{-g} \delta g^{\mu\nu}} &= \frac{1}{2} g_{\mu\nu} [-a_0^2 f_y + g^{\rho\sigma} (\partial_\rho \xi \partial_\sigma \phi + \partial_\rho \psi \partial_\sigma \chi) + 2\xi u^\rho u^\sigma R_{\rho\sigma} - \psi] \\ &\quad + \partial_\mu \phi \partial_\nu \phi f'_y - \partial_{(\mu} \xi \partial_{\nu)} \phi - \partial_{(\mu} \psi \partial_{\nu)} \chi - 2\xi [2u_{(\mu} u^\alpha R_{\nu)\alpha} + u_\mu u_\nu u^\alpha u^\beta R_{\alpha\beta}] \\ &\quad - [\square(\xi u_\mu u_\nu) + g_{\mu\nu} D_\alpha D_\beta (\xi u^\alpha u^\beta) - 2D_\alpha D_{(\mu} (\xi u_{\nu)} u^\alpha)]. \end{aligned} \quad (17)$$

In this and subsequent expressions we follow the usual convention in which parenthesized indices are symmetrized. Also, we denote the covariant derivative with respect to  $x^\mu$  by the symbol  $D_\mu$ .

It remains to specify the various scalars as nonlocal functionals of the metric. This follows from applying retarded boundary conditions to the field equations which result from varying (16),

$$\frac{16\pi G \delta\Delta S_y}{\sqrt{-g} \delta \xi} = \square \phi - 2u^\alpha u^\beta R_{\alpha\beta}, \quad (18)$$

$$\frac{16\pi G \delta\Delta S_y}{\sqrt{-g} \delta \psi} = \square \chi + 1, \quad (19)$$

$$\frac{16\pi G \delta\Delta S_y}{\sqrt{-g} \delta \phi} = \square \xi - 2D_\mu \left[ D^\mu \phi f'_y \left( \frac{g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi}{a_0^2} \right) \right], \quad (20)$$

$$\frac{16\pi G \delta\Delta S_y}{\sqrt{-g} \delta \chi} = \square \psi - 4D_\mu \left[ \frac{\xi g_\perp^{\mu\rho} u^\sigma R_{\rho\sigma}}{\sqrt{-g^{\alpha\beta} \partial_\alpha \chi \partial_\beta \chi}} \right]. \quad (21)$$

(Note the induced metric  $g_\perp^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$  which appears in equation (21) for  $\psi$ .) Solving for each of the four scalars involves inverting the scalar d’Alembertian  $\square$ , which would ordinarily allow us to freely specify each scalar and its first time derivative on the initial value surface. Permitting those degrees of freedom would result in two scalar ghosts [38,40]. The original nonlocal model is recovered by setting each scalar and its first time

derivative to zero on the initial value surface, which also eliminates the ghosts (and in fact all the modes associated with the scalars).

### B. Specialization to FLRW

On scales of 100 Mpc and larger the geometry of our Universe is well described by a homogeneous, isotropic and spatially flat metric in comoving coordinates,

$$ds_{\text{FLRW}}^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\vec{x} \cdot d\vec{x}. \quad (22)$$

The function  $a(t)$  is known as the scale factor and its logarithmic time derivative gives the Hubble parameter  $H(t)$ ,

$$H(t) \equiv \frac{\dot{a}}{a}. \quad (23)$$

The nonvanishing components of the affine connection are

$$\Gamma^i_{j0} = H\delta^i_j, \quad \Gamma^0_{ij} = Hg_{ij}. \quad (24)$$

This implies the following components of the curvature,

$$R^0_{i0j} = (\dot{H} + H^2)g_{ij}, \quad R^i_{jke} = H^2(\delta^i_k g_{je} - \delta^i_e g_{jk}), \quad (25)$$

$$\begin{aligned} R_{00} &= -3(\dot{H} + H^2), & R_{ij} &= (\dot{H} + 3H^2)g_{ij}, \\ R &= 6\dot{H} + 12H^2. \end{aligned} \quad (26)$$

The nonvanishing components of the second covariant derivative of a scalar  $S(t)$  are simple,

$$D_0 D_0 S = \ddot{S},$$

$$D_i D_j S = -H \dot{S} g_{ij} \Rightarrow \square S = -(\ddot{S} + 3H\dot{S}) = -\frac{1}{a^3} \frac{d}{dt} a^3 \dot{S}. \quad (27)$$

Of course the final expression for  $s \equiv \square S$ , with our retarded boundary conditions, results in a simple form for  $S = \frac{1}{\square} s$ ,

$$\left[ \frac{1}{\square} s \right] (t) = - \int_{t_i}^t \frac{dt'}{a^3(t')} \int_{t_i}^{t'} dt'' a^3(t'') s(t''). \quad (28)$$

We also require various contractions of double covariant derivatives of a second rank tensor whose nonzero components are restricted by homogeneity and isotropy to be  $T_{00}(t)$  and  $T_{ij} = T(t)g_{ij}$ . [Note that this  $T$  does not mean the trace of  $T_{\mu\nu}$ , and notably that it will vanish below for  $T_{\mu\nu} = \xi u_\mu u_\nu$ .] Some tedious but straightforward manipulations reveal

$$\square T_{00} = -\ddot{T}_{00} - 3H\dot{T}_{00} + 6H^2(T_{00} + T), \quad (29)$$

$$\square T_{ij} = [-\ddot{T} - 3H\dot{T} + 2H^2(T_{00} + T)]g_{ij}, \quad (30)$$

$$D_\alpha D_0 T^\alpha_0 = -\ddot{T}_{00} - 3H(\dot{T}_{00} + \dot{T}) + 3H^2(T_{00} + T), \quad (31)$$

$$D_\alpha D_i T^\alpha_j = [H\dot{T}_{00} + (\dot{H} + 4H^2)(T_{00} + T)]g_{ij}, \quad (32)$$

$$D_\alpha D_\beta T^{\alpha\beta} = \ddot{T}_{00} + 3H(2\dot{T}_{00} + \dot{T}) + 3(\dot{H} + 3H^2)(T_{00} + T). \quad (33)$$

The various auxiliary fields take simple forms when specialized to the FLRW geometry (22),

$$\phi(t) = 6 \int_{t_i}^t \frac{dt'}{a^3(t')} \int_{t_i}^{t'} dt'' a^3(t'') [\dot{H}(t'') + H^2(t'')], \quad (34)$$

$$\chi(t) = \int_{t_i}^t \frac{dt'}{a^3(t')} \int_{t_i}^{t'} dt'' a^3(t'') \Rightarrow u^\mu(t) = \delta^\mu_0, \quad (35)$$

$$\xi(t) = 2 \int_{t_i}^t dt' \dot{\phi}(t') f'_y \left( -\frac{\dot{\phi}^2(t')}{a_0^2} \right), \quad (36)$$

$$\psi(t) = 0. \quad (37)$$

Strictly speaking,  $u^\mu$  is ill defined on the initial value surface, because Eq. (8) is singular when  $\dot{\chi} = 0$ , but we can take the limit of the well-defined  $u^\mu(t)$  for  $t \rightarrow t_i$ . Alternative definitions of this timelike unit vector may also be chosen, like Eqs. (20)–(22) of Ref. [41].

Of course homogeneity and isotropy imply that any second rank tensor such as  $\Delta G_{\mu\nu}$  has only two distinct components when specialized to the FLRW geometry (22). We find them to be

$$\left. \frac{16\pi G \delta \Delta S_y}{\sqrt{-g} \delta g^{00}} \right|_{\text{FLRW}} = \frac{a_0^2}{2} f_y \left( \frac{-\dot{\phi}^2}{a_0^2} \right) + 3H\dot{\xi} + 6H^2\xi, \quad (38)$$

$$\left. \frac{16\pi G \delta \Delta S_y}{\sqrt{-g} \delta g^{ij}} \right|_{\text{FLRW}} = - \left[ \frac{a_0^2}{2} f_y \left( \frac{-\dot{\phi}^2}{a_0^2} \right) + \ddot{\xi} + \left( \frac{\dot{\phi}}{2} + 4H \right) \dot{\xi} + (4\dot{H} + 6H^2)\xi \right] g_{ij}. \quad (39)$$

### C. The reconstruction problem

If the function  $f_y(Z)$  in expression (14) were known for  $Z < 0$  then one would add expression (38) to the usual Friedmann equation and solve for the scale factor  $a(t)$ ,

$$3H^2 + \left\{ \frac{a_0^2}{2} f_y \left( \frac{-\dot{\phi}^2}{a_0^2} \right) + 3H\dot{\xi} + 6H^2\xi \right\} = 8\pi G\rho, \quad (40)$$

where  $\rho$  describes all matter sources, including radiation and baryonic matter, but not dark matter nor dark energy which would be reproduced by the nonlocal terms within the curly brackets. However, MOND phenomenology only determines the asymptotic forms of  $f_y(Z)$  for  $0 < Z \lesssim 1$  and for  $Z \gg 1$ ,

$$0 < Z \lesssim 1 \Rightarrow f_y(Z) = \frac{1}{2}Z - \frac{1}{6}Z^3 + O(Z^2), \quad (41)$$

$$1 \ll Z < \infty \Rightarrow f_y(Z) \rightarrow 0. \quad (42)$$

The reconstruction problem consists of instead regarding  $a(t)$  as known—along with how the energy density  $\rho$  depends upon  $a(t)$ —and then solving the modified Friedmann equation (40) to find the function  $f_y(Z)$  which supports the desired expansion history (similarly to what has been done in scalar-tensor theories [42] or other nonlocal models [43]).

Once the reconstruction problem has been solved the model is fixed, and one can subject it to meaningful tests by working out its predictions for the growth of cosmological perturbations. Many modified gravity models have been analyzed in this way. For example, the free function  $f(\frac{1}{\square}R)$  of “nonlocal cosmology” [37,40] was determined (numerically) to support the  $\Lambda$ CDM expansion history, without a cosmological constant [43], then its predictions for structure formation were shown to be in conflict with the most recent data on weak lensing and redshift space distortions [44]. In this subsection we will derive a second order, linear differential equation for  $f_y(Z)$  which could be numerically solved to support a given expansion history.

The first problem with (40) is that time is the natural variable, rather than  $Z(t) \equiv -[\dot{\phi}(t)/a_0]^2$ . We therefore employ the new symbol  $f(t)$  to regard the dependent variable as a function of time,

$$f(t) \equiv f_y \left( \frac{-\dot{\phi}^2(t)}{a_0^2} \right) \Rightarrow f'_y \left( \frac{-\dot{\phi}^2}{a_0^2} \right) = -\frac{a_0^2 \dot{f}}{2\dot{\phi}\ddot{\phi}}. \quad (43)$$

The second problem is that the auxiliary scalar  $\xi$  given by expression (36) involves an integral. We therefore divide (40) by  $3a_0^2 H^2(t)$  and differentiate,

$$\frac{d}{dt} \left[ -\frac{\dot{f}}{H\dot{\phi}} + \frac{f}{6H^2} \right] - \frac{2\dot{f}}{\dot{\phi}} = \frac{d}{dt} \left[ \frac{8\pi G\rho}{3a_0^2 H^2} \right]. \quad (44)$$

Equation (44) is a linear, second order differential equation for  $f(t)$  which can be evolved forward from  $t = t_i$ , using the explicit expression (34) for  $\phi(t)$  in terms of the known expansion history. From the mathematical point of view,  $f(t)$  is fully determined from the two initial conditions  $f(t_i) = 0$  and  $f'(t_i) = 2[8\pi G\rho(t_i) - 3H(t_i)^2]/a_0^2$ , implied by Eqs. (34), (36), (40) and (43). However, we actually have more freedom because  $H(t)$  is not known with infinite precision. At early times during radiation domination, we only need our nonlocal terms, within the curly brackets of Eq. (40), to be negligible with respect to  $8\pi G\rho_{\text{radiation}}$ . As will be detailed in a forthcoming publication, it is thus possible to integrate (44) backwards in time, starting from the present epoch, while still integrating forward (34) and (36) to respect the crucial constraints  $\phi(t_i) = \dot{\phi}(t_i) = \xi(t_i) = \dot{\xi}(t_i) = 0$  which eliminate ghost excitations. One of the two integration constants in the solution of (44) is then fixed by requiring that the undifferentiated equation (40) holds, while the second constant allows us to match the limit of  $f_y(Z)$  for  $Z \rightarrow 0^-$  to the needed MOND form (9) for  $Z > 0$ .

The final step is inverting the relation between  $Z$  and  $t$  implied by relation (34) to solve for  $t$  as a function of  $Z$ ,

$$Z(t) = -\left[ \frac{6}{a_0 a^3(t)} \int_{t_i}^t dt' a^3(t') [\dot{H}(t') + H^2(t')] \right]^2 \Rightarrow t(Z). \quad (45)$$

The auxiliary fields  $\theta$  and  $\omega$  are determined by applying retarded boundary conditions to the equations which derive from varying  $\Delta S_x$ ,

$$\frac{16\pi G}{\sqrt{-g}} \frac{\delta \Delta S_x}{\delta \omega} = \square \theta - \left[ u^\alpha u^\beta R_{\alpha\beta} - \frac{1}{2} R \right], \quad (50)$$

The desired function is  $f_y(Z) = f(t(Z))$ . Except for very simple expansion histories this analysis will need to be done numerically.

### III. MODELS WHICH INCLUDE $X$ WITH CONSTANT $a_0$

The purpose of this section is to work out how  $\Delta G_{\mu\nu}$  changes if, in addition to the mandatory MOND term (14) we elect to also add the optional term,

$$\Delta \mathcal{L}_x = \frac{1}{16\pi G} \times a_0^2 f_x \left( \frac{X[g]}{a_0^2} \right) \sqrt{-g}, \quad (46)$$

where  $X[g]$  is the nonlocal invariant defined by expressions (10), (8) and (12), and  $a_0$  is strictly constant. Much of the analysis is similar to what was done in Sec. II for  $\Delta \mathcal{L}_y$ . In particular, it is again useful to localize the system using auxiliary scalar fields. In addition to  $\chi$  and  $\psi$  which are already present in the mandatory term (16)—we require a scalar  $\theta$  which bears the same relation to  $X[g]$  that  $\phi$  bears to  $Y[g]$ ,

$$\theta \longrightarrow \frac{1}{\square} \left[ u^\alpha u^\beta R_{\alpha\beta} - \frac{1}{2} R \right]. \quad (47)$$

Of course we also need a Lagrange multiplier field  $\omega$  to enforce this relation. This makes the local version,

$$\begin{aligned} \Delta \mathcal{L}_x = \frac{1}{16\pi G} \left\{ a_0^2 f_x \left( \frac{g^{\mu\nu} \partial_\mu \theta \partial_\nu \theta}{a_0^2} \right) - \partial_\mu \omega \partial_\nu \theta g^{\mu\nu} \right. \\ \left. - \omega \left[ u^\alpha u^\beta R_{\alpha\beta} - \frac{1}{2} R \right] \right\} \sqrt{-g}. \end{aligned} \quad (48)$$

We remind the reader that  $u^\mu$  is the normalized gradient (8) of  $\chi$ , which appears, along with its Lagrange multiplier  $\psi$ , in the mandatory term (16).

The correction (48) makes to  $\Delta G_{\mu\nu}$  is

$$\begin{aligned} \frac{16\pi G}{\sqrt{-g}} \frac{\delta \Delta S_x}{\delta g^{\mu\nu}} = \frac{1}{2} g_{\mu\nu} \left[ -a_0^2 f_x + \partial_\rho \omega \partial_\sigma \theta g^{\rho\sigma} + \omega \left( u^\alpha u^\beta R_{\alpha\beta} - \frac{1}{2} R \right) + \partial_\mu \theta \partial_\nu \theta f'_x \right. \\ \left. - \partial_{(\mu} \omega \partial_{\nu)} \theta - \omega \left[ 2u_{(\mu} u^\alpha R_{\nu)\alpha} + u_\mu u_\nu u^\alpha u^\beta R_{\alpha\beta} - \frac{1}{2} R_{\mu\nu} \right] + \frac{1}{2} (g_{\mu\nu} \square - D_\mu D_\nu) \omega \right. \\ \left. - \frac{1}{2} [\square (\omega u_\mu u_\nu) + g_{\mu\nu} D_\alpha D_\beta (\omega u^\alpha u^\beta) - 2D_\alpha D_{(\mu} (\omega u_{\nu)} u^\alpha)] \right]. \end{aligned} \quad (49)$$

$$\frac{16\pi G}{\sqrt{-g}} \frac{\delta \Delta S_x}{\delta \theta} = \square \omega - 2D_\mu \left[ D^\mu \theta f'_x \left( \frac{g^{\rho\sigma} \partial_\rho \theta \partial_\sigma \theta}{a_0^2} \right) \right]. \quad (51)$$

The equation for the auxiliary field  $\chi$  is still (19), but the equation for  $\psi$  receives contributions from the  $\chi$  dependence (through  $u^\mu$ ) in both  $\Delta \mathcal{L}_y$  and  $\Delta \mathcal{L}_x$ ,

$$\frac{16\pi G \delta \Delta S_{x+y}}{\sqrt{-g} \delta \chi} = \square \psi - 2D_\mu \left[ \frac{(2\xi + \omega) g_\perp^{\mu\rho} u^\sigma R_{\rho\sigma}}{\sqrt{-g^{\alpha\beta} \partial_\alpha \chi \partial_\beta \chi}} \right]. \quad (52)$$

Just as for the mandatory addition  $\Delta \mathcal{L}_y$ , we do not regard the auxiliary scalars as fundamental fields with arbitrary initial value data. That would result in the combination  $\theta - \omega$  being a ghost [38,40]. We instead define each auxiliary field and its first time derivative to vanish at  $t = t_i$ .

Specializing to the FLRW geometry is a straightforward extension of the analysis of subsection II B. The addition of the optional term (46) does not change the FLRW expressions (34)–(37) for the four auxiliary scalars  $\phi$ ,  $\chi$ ,  $\xi$  and  $\psi$  of the mandatory MOND term (14). [Note in particular that  $\psi$  still vanishes identically, Eq. (37), because  $g_\perp^{\mu\rho} u^\sigma R_{\rho\sigma} = g_\perp^{\mu\rho} R_{\rho 0} = g_\perp^{\mu 0} R_{00} = 0$  in FLRW.] The two new auxiliary scalars become

$$\theta(t) = \int_{t_i}^t \frac{dt'}{a^3(t')} \int_{t_i}^{t'} dt'' a^3(t'') [6\dot{H}(t'') + 9H^2(t'')], \quad (53)$$

$$\omega(t) = 2 \int_{t_i}^t dt' \dot{\theta}(t') f_x' \left( -\frac{\dot{\theta}^2(t')}{a_0^2} \right). \quad (54)$$

And the contributions to the two nonzero components of  $\Delta G_{\mu\nu}$  are

$$\frac{16\pi G \delta \Delta S_x}{\sqrt{-g} \delta g^{00}} \Big|_{\text{FLRW}} = \frac{a_0^2}{2} f_x \left( \frac{-\dot{\theta}^2}{a_0^2} \right) + 3H\dot{\omega} + \frac{9}{2} H^2 \omega, \quad (55)$$

$$\begin{aligned} \frac{16\pi G \delta \Delta S_x}{\sqrt{-g} \delta g^{ij}} \Big|_{\text{FLRW}} &= - \left[ \frac{a_0^2}{2} f_x \left( \frac{-\dot{\theta}^2}{a_0^2} \right) + \ddot{\omega} + \left( \frac{\dot{\theta}}{2} + 3H \right) \dot{\omega} \right. \\ &\quad \left. + \left( 3\dot{H} + \frac{9}{2} H^2 \right) \omega \right] g_{ij}. \end{aligned} \quad (56)$$

An important observation is that  $X$  vanishes for exact matter domination,  $H(t) = \frac{2}{3t}$ . This means that the optional correction cannot have much effect on cosmology at the time of recombination, or on the early stages of structure formation. It also means that the optional correction cannot supply the MOND enhancement of gravity which would be necessary to compensate for the absence of dark matter at early times.

#### IV. MAKING $a_0$ DYNAMICAL

MOND phenomenology only constrains the function  $f_y(Z)$  of the mandatory MOND addition (14) for  $Z > 0$ . Based on subsection II C, it seems possible to adjust how  $f_y(Z)$  behaves for  $Z < 0$  to support an arbitrary expansion history. However, the variable  $Y[g]$  is essentially  $-H^2(t)$  for cosmology, whereas  $a_0 \sim H_0/2\pi$ , so the argument

$Z = Y/a_0^2 \sim -4\pi^2 H^2/H_0^2$  varies enormously over interesting cosmological events such as nucleosynthesis ( $Z \sim -10^{32}$ ) and recombination ( $Z \sim -10^{10}$ ). This raises concerns about fine tuning. These concerns can be ameliorated by making the MOND acceleration  $a_0$  some functional  $\alpha[g]$  of the metric so that it changes with the scale of cosmological acceleration. There are many, many plausible choices for  $\alpha[g]$ . To develop some quantitative understanding of the consequences of a sliding scale, we here work out the effect of a simple choice (13) in which  $a_0$  is replaced by  $1/6\pi$  times the expansion, i.e.,  $D_\mu u^\mu/6\pi$  where the timelike four-velocity  $u^\mu[g]$  is defined in Eqs. (8) and (12). Because the optional MOND addition (46) does not seem to have much effect for cosmology we only derive results for the mandatory addition (14).

The replacement (13) causes only three changes in the general metric field equations (17) and (18)–(21). The first and most obvious change is that the factors of  $a_0$  in (17) and (20) get replaced by  $\alpha[g]$ . The second change is that the addition to the Einstein tensor acquires an extra contribution from the metric dependence of  $\alpha[g]$ ,

$$\begin{aligned} (\Delta G_{\mu\nu})_{\text{new}} &= (\Delta G_{\mu\nu})_{\text{old}} + g_{\mu\nu} [\alpha^2 f_y - g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi f_y'] \\ &\quad + \frac{1}{6\pi} [g_{\mu\nu} u^\gamma \partial_\gamma - 2u_{(\mu} \partial_{\nu)} - u_\mu u_\nu u^\gamma \partial_\gamma] \\ &\quad \times \left[ \alpha f_y - \frac{1}{\alpha} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi f_y' \right]. \end{aligned} \quad (57)$$

The final change is that equation (21) for the auxiliary scalar  $\psi$  picks up an extra term from the  $\chi$  dependence of  $\alpha$ ,

$$\begin{aligned} \frac{16\pi G \delta \Delta S_y}{\sqrt{-g} \delta \chi} &= \square \psi - D_\mu \left[ \frac{4\xi g_\perp^{\mu\nu} u^\rho R_{\rho\nu} + \frac{1}{3\pi} g_\perp^{\mu\nu} \partial_\nu [\alpha f_y - \frac{1}{\alpha} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi f_y']}{\sqrt{-g^{\kappa\lambda} \partial_\kappa \chi \partial_\lambda \chi}} \right]. \end{aligned} \quad (58)$$

These small alterations in the functional form of the field equations conceal vast changes in their numerical values. That becomes apparent upon specialization to the FLRW cosmology (22). In this case the functional  $\alpha[g]$  becomes

$$\alpha[g]_{\text{FLRW}} = \frac{H(t)}{2\pi}. \quad (59)$$

The auxiliary scalars  $\phi$ ,  $\chi$  and  $\psi$  are unchanged from expressions (34), (35) and (37), respectively, but  $\xi$  becomes

$$\xi(t) = 2 \int_{t_i}^t dt' \dot{\phi}(t') f_y' \left( \frac{-4\pi^2 \dot{\phi}^2(t')}{H^2(t')} \right). \quad (60)$$

Our nonlocal addition to the Friedmann equation is

$$\Delta G_{00} = -\frac{H^2}{8\pi^2} f_y - \dot{\phi}^2 f_y' + 3H\dot{\xi} + 6H^2\xi, \quad (61)$$

where each of the functions  $f_y(Z)$  is evaluated at  $Z = -4\pi^2\dot{\phi}^2/H^2$ .

Let us underline two subtleties related to the way we introduce a time-dependent  $a_0$  through the replacement  $a_0 \rightarrow \alpha[g]$ . First of all, in a FLRW background, the argument  $Z$  of the function  $f_y(Z)$  will remain of the order of  $-4\pi^2$  at all times, so that the reconstruction of an arbitrary expansion history seems more difficult to achieve. This needs to be analyzed numerically. There are however many other possible definitions of a time-dependent  $a_0$ , and significant but not-too-large variations of  $Z$  are possible for instance with some kind of geometrical mean between  $\alpha[g]$  and the constant  $a_0$ . Independently of the cosmological reconstruction, corresponding to  $Z < 0$ , note that a time-dependent  $a_0 \rightarrow \alpha[g]$  will be quite useful in the MOND correction (9) for  $Z > 0$ , so that this modified dynamics happen for larger accelerations at earlier times, mimicking the clustering effects of dark matter.

The second subtlety is that our definition (13) for  $\alpha[g]$  is likely to vanish within gravitationally bound systems, because the local expansion  $D_\mu u^\mu$  should not keep any information about the asymptotic cosmological evolution (although this needs to be confirmed by further examination). This would force the model always into the general relativistic regime, which would turn off the MOND force even in the static, spherically symmetric and ultraweak field regime. One might deal with this by simply using  $a_0 + \alpha[g]$  as the acceleration scale [45] entering our nonlocal action, or one might devise a more nonlocal version of  $\alpha[g]$  whose value inside a gravitationally bound system depends upon the cosmological expansion around it [46].

## V. CONCLUSIONS

Our previous work on extending MOND to a relativistic, metric theory led to consideration of two nonlocal scalar functionals of the metric: a mandatory one  $Y[g]$  given in expression (7) and optional one  $X[g]$  given in expression (10) [28]. To recover MOND with sufficient weak lensing requires that the  $Y$  term, and allows that the  $X$  term, be added to the gravitational Lagrangian in the form

$$\begin{aligned} \frac{\Delta\mathcal{L}}{\sqrt{-g}} &= \frac{a_0^2}{16\pi G} \left[ f_y \left( \frac{Y[g]}{a_0^2} \right) + f_x \left( \frac{X[g]}{a_0^2} \right) \right] \\ &= \frac{1}{32\pi G} \left[ (Y - X) - \frac{(Y^{\frac{3}{2}} - X^{\frac{3}{2}})}{3a_0} + \dots \right]. \end{aligned} \quad (62)$$

Our previous study gave the field equations for static, spherically symmetric geometries in the ultraweak field limit. In this paper we have derived the field equations—expressed as an addition  $\Delta G_{\mu\nu}$  to the usual Einstein tensor—for arbitrary functions  $f_y$  and  $f_x$  and for an arbitrary metric. Our result for  $\Delta G_{\mu\nu}$  from the mandatory term is equation (17), with auxiliary fields (18)–(21). Our result for  $\Delta G_{\mu\nu}$  from the optional term is equation (49), with auxiliary fields (50)–(52).

We also specialized the general field equations to the FLRW geometry (22) of cosmology. Our results for the mandatory term are relations (34)–(39); for the optional term they are relations (53)–(56). Because  $X$  happens to vanish for a matter-dominated cosmology, the optional term does not seem likely to play much role in cosmology. However, in subsection II C we described a technique by which the free function  $f_y(Z)$  could be constructed for  $Z < 0$ —which is not constrained by MOND phenomenology—to support a general expansion history  $a(t)$ .

Although the reconstruction problem can be solved for the mandatory term, there will be large variations in the argument  $Z = -\dot{\phi}^2/a_0^2 \sim -4\pi^2 H^2/H_0^2$  over the course of cosmological history. This makes it likely that the extra MOND force—which is needed at early times if there is no dark matter—will only become effective at recent times. That argues for making  $a_0$  dynamical. In Sec. IV we derived the field equations and their specialization to cosmology for a simple ansatz (13) in which the MOND constant  $a_0$  changes with the expansion of the Universe. One obvious problem is that our definition (13) for the dynamical MOND acceleration  $\alpha[g]$  probably vanishes inside a gravitationally bound structure, so that the MOND force would always be turned off. We conclude that a more nonlocal ansatz may be necessary, in which the MOND acceleration inside gravitationally bound structures can still be determined by the cosmological expansion around them.

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