On the shoulders of Laplace

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Abstract

In 1799, Laplace derived the system of differential equations (now called Liouville-Euler) that fully describes the motions of the rotation axis of any celestial body. Laplace showed that only the gravitational forces and kinetic moments from other celestial bodies influence the rotation of any one of them. The equations involve three Euler angles that specify the motions of a body's rotation axis; they can be reduced to a system of two equations for the inclination and time derivative of the declination of the rotation axis. Laplace showed the existence of a forced annual oscillation and the so-called free Chandler wobble. Most current theories retain only two Euler angles and invoke an elastic Earth to match observations. We analyze the much longer time series of polar motion (coordinates m_1 and m_2 of the rotation pole at the Earth's surface) now available, in order to further explore phenomena that Laplace could not investigate, given the dearth of data in his time. We use singular spectral analysis (SSA) to extract components of the time series. The first three components (trend or Markowitz drift, forced annual oscillation and free Chandler oscillation) account for 73% of the variance of polar motion. Under the current theory, their modulation is thought to be a response to reorganization of oceanic and atmospheric masses. However, the periods of the first six SSA components of polar motion have been encountered in previous studies of sunspots and in the ephemerids of Jovian planets. We also analyze the derivatives of the envelopes of the three SSA components of polar motion. Again, most of these components have periods and modulations that correspond to the ephemeris (periods and combinations of commensurable periods) of Jovian planets. Examples include 171.5 yr (the Jose cycle linked to Neptune), 90 yr (the Gleissberg cycle linked to Uranus), 40 yr (a commensurable period linked to the Jovian planets), 22 yr, 11 yr (Jupiter, Sun), 60 yr, 30 yr (Saturn). Figure 3 can be considered as the central result of the paper. It shows that the sum of forces of the four Jovian planets matches in a striking way the polar motion reconstructed with SSA components (the Markowitz trend removed). All our results argue that significant parts of Earth's polar motion are a consequence (rather than a cause) of the evolution of planetary ephemerids. The Sun's activity and many geophysical indices show the same signatures, including many climate indices. Two different mechanisms (causal chains) are likely at work: a direct one from the Jovian Planets to Earth, another from planetary motions to the solar dynamo; variations in solar activity would in turn influence meteorological and climatic phenomena. Given the remarkable coincidence between the quasi-periods of many of these phenomena, it is reasonable to assume that both causal chains are simultaneously at work. In that sense, it is not surprising to find the signatures of the Schwabe, Hale and Gleissberg cycles in many

terrestrial phenomena, reflecting the characteristic periods of the combined motions of the Jovian planets.

1 - Introduction

On July 5 1687, Isaac Newton published the three volumes of his *Principia Mathematica*, in which he put on a firm ground the law of universal attraction and the general laws of mutual attraction of masses. In the following two centuries, a corpus of laws that explained the motions of celestial bodies was established and vindicated by observations. Foremost among these works, Laplace published his *Traité de Mécanique Céleste* (Treatise of Celestial Mechanics) in 1799.

Based on Newton's law and the fundamental principle of dynamics, he established the general equations that govern the motions of material bodies (Laplace, 1799, book 1, chapter 7, page 74, system (D)). This system of differential equations of first order was later given the names of Liouville and Euler. It establishes both the rotation and translation of the rotation axis of any celestial body, and in particular the Earth. These same equations can be found in Guinot (in Coulomb and Jobert, 1977, p. 530) and more recently in the reference book of Lambeck (2005, p. 31). They are recalled in Appendices 1 and 2 in their most general form. When the forces and the moments that act on Earth are taken to be zero (i.e. the right hand side of the equations is zero), the solution for the axis is a free oscillation with a Euler period $1/\sigma$ of 306 days (using the known values of the mean angular velocity and axial and equatorial moments of inertia $(\sigma=((C-A)/A)\Omega)$). Based on observations made between June 1884 and November 1885, Chandler (1891a,b) obtained a value of 427 days for $1/\sigma$. Data provided by the *International Earth Rotation and Reference System Service* (IERS) yield a $1/\sigma$ that has varied between 431 days in 1846 and 434 days in 2020.

Newcomb (1892) verified Chandler's observations and concluded that the Earth should be viewed as an elastic body submitted to oceanic stresses. For this, Love numbers were introduced (Love, 1909). As a result, the Liouville-Euler system (D) of Laplace was made less general. Hough (1895) reinforced the idea that what made the Chandler period 121 days longer than the "theoretical" value was the fact that the Earth behaved as an elastic body. Based on Poincaré's (1885) work on the stability of rotating fluids with a free surface, Hough showed that the period should decrease rather than increase if one did not take elasticity into account. Works in the following decades strengthened the notion that the fluid envelopes of Earth (ocean, atmosphere and mantle) acted on Earth's rotation axis. An increasingly precise theory was thus proposed, whereas observations seemed to be increasingly remote from predictions.

Two papers (Peltier and Andrews, 1976; Nakiboglu and Lambeck, 1980) further strengthened the theory of an elastic Earth whose rotation axis was influenced by both its internal and external fluid envelopes. An important concept was that of Global Isostatic Adjustment (GIA), in which the Earth has a visco-elastic response to stress (load) variations, that originated at the onset of the last ice age. Melting ice would lead to sea level rise and a reorganization of surface masses that eventually modified the inclination of the rotation pole. Rather than writing in a physically explicit way the forces implied in system (D), as done by Laplace and Poincaré, more or less complex “excitation functions” were introduced (Appendix 1).

We return to the founding work of Laplace (1799) to see how these problems can be tackled further. In what follows, we refer to volumes, chapters, pages and equation numbers in the original edition of the *Traité de Mécanique Céleste*. Throughout the *Treatise*, Laplace (1799) rigorously shows that, whatever the nature of the oceans and atmosphere, the only thing that influences the rotation of celestial bodies is the action of other celestial bodies. On page 347 (chapter 1, volume 5) Laplace (1799) writes (this quotation is given in the original French in Appendix 4): “*We have shown that the mean rotation movement of Earth is uniform, assuming that the planet is entirely solid and we have just seen that the fluidity of the sea and of the atmosphere should not alter this result. It would seem that the motions that are excited by the Sun’s heat, and from which the easterly winds are born should diminish the Earth’s rotation: these winds blow between the tropics from west to east and their continued action on the sea, on the continents and on the mountains they encounter, should seem to weaken imperceptibly that rotation movement. But the principle of conservation of areas, shows to us that the total effect of the atmosphere on this movement must be insensible; for the solar heat in dilating equally the air in all directions, should not alter the sum of areas covered by the vector radii of each molecule of the Earth and of the atmosphere, and when multiplied respectively by the corresponding molecules; which requires that the rotation motion be not diminished. We are therefore assured that as the winds analyzed diminish this motion, the other movements of the atmosphere that occur beyond the tropics, accelerate it by the same amount. One can apply the same reasoning to earthquakes, and in general, to all that can shake the Earth in its interior and at its surface. Only the displacement of these parts can alter this motion; if, for instance a body placed at the pole, was transported to the equator; since the sum of areas must always remain the same, the earth’s motion would be slightly diminished; but for it to be noticeable, one should suppose the occurrence of great changes in the Earth’s constitution.*”

These views are also shared by Poincaré (1899). They seem to be different from modern views as synthesized for instance by Lambeck (2005). These authors agree on the Liouville-Euler system (D for Laplace) of differential equations, but the forces that act on the Earth are different (and interpreted in a different way, as shown below). In the present paper, we attempt to check Laplace's full system using the observations that have accumulated and improved since Laplace's time (time series starting in 1750 for the oldest and no later than 1850 for the shortest ones).

We first discuss some of the core ideas of the paper, based on Laplace's original developments (section 2). We then recall some concepts and tools that we use in the paper and introduce the data, i.e. the coordinates of the Earth's rotation pole from 1846 to 2020 (section 3). In section 4, we establish a striking result that is central to the paper: the detrended polar motion is highly correlated with the sum of the forces exerted by the four Jovian planets. We next submit the data to Singular Spectral Analysis (SSA) and discuss the first SSA components (section 4): the Markowitz, annual and Chandler rotations. Then, in section 5, we discuss the SSA components of the derivatives of the three components above. In section 6, we give several other examples, such as the excellent correlation of the 40yr SSA component of the derivative of the envelope of the Chandler oscillation with the 40yr SSA component of the combined forces of Uranus and Neptune. We end with a discussion and concluding remarks (section 7).

2 –Forces, Moments and the Liouville-Euler System of Equations

In most classical applications of the mechanics of planetary rotation, one uses only the first two components of the trio of Euler angles, i.e. the coordinates of the rotation pole at the Earth's surface (Figure A1, Appendix 1). The Earth rotates about the Sun (and so do the other 7 planets) in the ecliptic plane that is almost perpendicular to the rotation axis. The Sun carries more than 99% of the mass of the solar system, and can be considered rather motionless (its center of gravity actually travels along a "small" variable "ellipse"). In addition to the gravitational attractions, one must consider the orbital kinetic moments of all planets (in others words the moment of the momentum, see eq. B1, Appendix 2), as emphasized by Laplace (1799). Planets carry more than 99% of the total angular momentum of the system (19.3, 7.8, 1.7 and 1.7 x 10^{42} kgm^2s^{-1} respectively for Jupiter, Saturn, Uranus and Neptune). This can be compared to the Sun's attraction at the Earth's orbit, 3.5 10^{22} $kg.m.s^{-2}$, that can be transformed to the dimension of a kinetic moment by multiplying it by the Sun-Earth distance and the orbital revolution period of Earth, yielding 1.7 10^{41} $kg.m^2.s^{-1}$: that is not negligible compared to the order of magnitude of the kinetic moments of the Jovian planets (to 1 or 2 orders of magnitude).

The central idea of this paper is to analyze variations in the Earth's rotation axis under the influence not only, as in many classic treatments (e.g. Dehant and Mathiews, 2015, ch. 2), of gravitational potentials, but also of kinetic moments. The classical system of differential equations that describe the pole's motion (Liouville-Euler) links the sum of simple physical entities with their time derivatives, hence a first order linear system (Appendix 1, part 1). See Bode (1945) for more on the definition and consequences that can be drawn from such linear systems. One is that causes and consequences are similar, up to a constant factor, if the system is not too dissipative and is maintained: this implies that gravitational potentials, kinetic moments (of Jovian planets) and polar motions should share characteristic features.

Polar motion is described by three coordinates, usually labeled m_1, m_2 and m_3 (Appendix 1). If one only wants to study the perturbations due to the gravitational potential of a planet in rotation about itself, two coordinates, m_1 and m_2, are sufficient to describe the motion. In the case of our Solar system, planets revolve about the Sun in (or close to) the ecliptic plane; the moments they generate are perpendicular to that plane (Appendix 2). They act on the *inclinations* of the rotation axes of all planets, including Earth's. This is the well-known phenomenon of interaction of spinning tops and is adequately described by the Liouville-Euler equations. This was known to Laplace who chose not to use the three Euler angles, but gave all the analytic formulas that allow one to compute the *inclination* θ of the rotation axis (Figure A1 and Appendix 3) as a function of time, under the influence of the Moon and Sun (Laplace, 1799; book 5, page 317, number 5), and the time derivative of the *declination* ψ of the rotation axis (Laplace, 1799; book 5, page 318, number 6). θ and ψ are defined in Laplace (1799; book 1, page 73, number 26). Laplace (1799; book 5, pages 352-355, number 14) deduces that, when neither the Moon nor the Sun act on Earth (conjunction nodes), the time derivative of the declination (which in modern terms is the Euler period $1/\sigma$) has a value of 306 days (Appendix 3). This value is fully determined by the Earth's moments of inertia (i.e. the internal distribution of masses).

The equations derived by Laplace are:

$$\theta = h + \frac{3m}{4n}.\left(\frac{2C-A-B}{C}\right).\left\{\begin{array}{c} \frac{1}{2}.sin(\theta).\left\{cos(2\nu) + \frac{\lambda m}{m'}.cos(2\nu')\right\} \\ -(1+\lambda).m.cos(\theta).\sum.\frac{c}{f}.cos(ft+\varsigma) \\ +\frac{\lambda c'}{f'}.cos(\theta).cos(f't+\varsigma') \end{array}\right.$$

$$\frac{d\psi}{dt} = \frac{3m}{4n}.(\frac{2C-A-B}{C}).\left\{\begin{array}{c} (1+\lambda).m.cos(\theta) - \frac{cos(\theta)}{2dt}.\left\{d.sin(2\nu)+\frac{\lambda m}{m'}.d.sin(2\nu')\right\} \\ +(1+\lambda).m.\frac{cos^2(\theta)-sin^2(\theta)}{sin(\theta)}.\sum .c.cos(ft+\varsigma) \\ +\lambda.m\frac{cos^2(\theta)-sin^2(\theta)}{sin(\theta)}.c'.cos(f't+\varsigma') \end{array}\right.$$

All celestial and terrestrial parameters in these equations are defined in Appendix 2. The time variation of declination of the Earth’s rotation pole is a function of inclination. Since the *(θ,ψ)* and *(m_1, m_2)* couples represent the same physics, the pattern of the sum of planetary kinetic moments that “force” part of the Earth’s polar motions should be found in m_1 and m_2 (see below). Laplace obtained these equations taking into account “only” the Moon and Sun.

When the Moon and Sun act with maximum effect (conjunction bellies) $1/\sigma$ reaches a value of 578 days. $1/\sigma$ therefore oscillates between 306 and 578 days; Chandler (1891) found a value of 427 days and today one observes values of 432-434 days. Both inclination θ and declination ψ drift.

3 –The Toolbox: Rotation Pole Data, Ephemerids, Commensurability and Singular Spectrum analysis

Some of the tools and data needed to pursue our goal are listed in this section. Of course, we require knowledge of planetary *ephemerids*, that are given by the IMCCE. Then we need/use:

3-1 Rotation pole data: Laplace did not have sufficient observations to demonstrate the influence of planets, though he certainly did not deny their possible role. We now have sufficiently long series of observations to test his full theory.

The rotation pole is defined by its components m_1 and m_2, respectively on the Greenwich (0°) and 90°E meridians (Figure A1). Two series of measurements of (m_1, m_2) are provided by IERS[1] under the codes EOP-C01-IAU1980 and EOP-14-C04. The first one runs from 1846 to July 1st 2020 with a sampling rate of 18.26 days, and the second runs from 1962 to July 1st 2020 with daily sampling (also giving access to the length of day). Figure 1 shows the components m_1 and m_2 of the longer series (data are given in milli arc second - mas - and converted here in radians per second - rad.s^{-1}). Figure 2 shows the Fourier spectrum of component m_1, corrected for a degree 2 non linear trend. The forced annual oscillation and the free Chandler oscillation with period close to 1.19 yr are conspicuous.

1 https://www.iers.org/IERS/EN/DataProducts/EarthOrientationData/eop.html

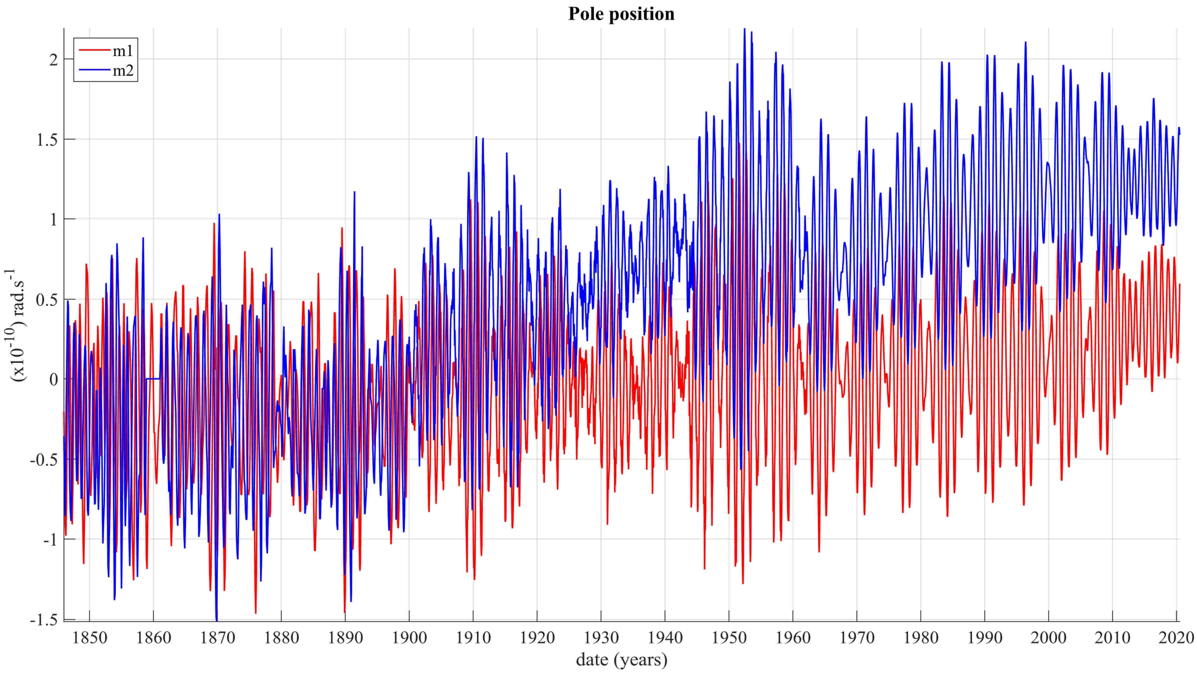

Figure 1: Components (m_1, m_2) of polar motion since 1846 (time series EOP-C01-IAU1980)

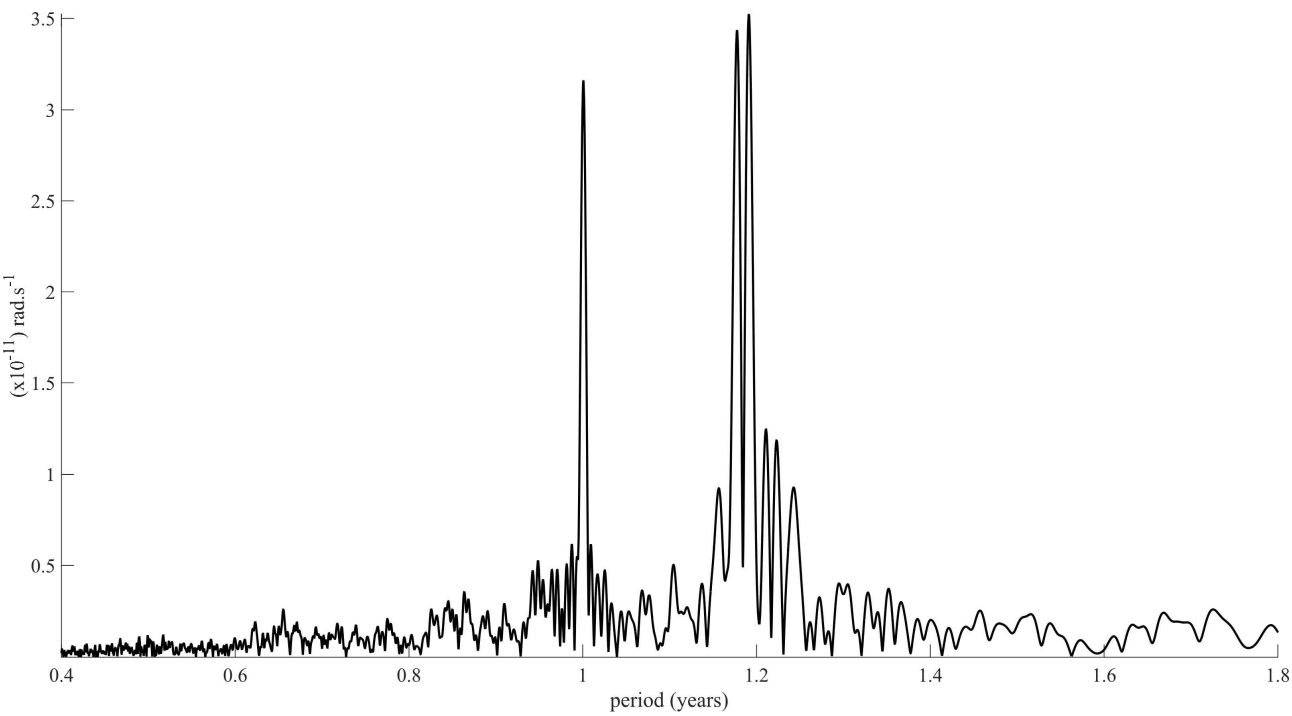

Figure 2: Fourier spectrum of component m_1, corrected for a degree 2 non linear trend

3-2 Commensurability: Then we use the concept of *commensurability*. One talks about commensurability when the ratio of the periods of 2 planets can be expressed as a fraction with integer numerator and denominator less than 9 (Mörth and Schlamminger, 1979; Okhlopkov, 2016; Scafetta, 2020). Planets encounter a resonance and can be paired, and each pair can be considered as a single object (an egregor or aggregate). Jupiter/Saturn and Uranus/Neptune form two pairs. Pairs of pairs can also be considered, thus the set (Jupiter/Saturn)/(Uranus/Neptune). Many analyses of sunspot series (Lassen and Friis-Christensen, 1995; Hataway, 2015; Usoskin et al., 2016; Le Mouël et al., 2017; Stefani et al., 2019; Courtillot et al., 2021; Le Mouël et al., 2020a; Stefani et al., 2020) and of a number of geophysical phenomena (Courtillot et al., 2013; Scafetta, 2016; Lopes et al., 2017; Scafetta et al., 2019; Bignami et al., 2020; Le Mouël et al., 2019a; Le Mouël et al., 2019b; Hilgen et al., 2020; Le Mouël et al., 2020b; Zaccagnino et al., 2020; Le Mouël et al., 2021) contain components with periods that can be attributed to Jovian planets to first order, and all planets including the telluric ones to second order (Courtillot et al, 2021). Table 1 lists planetary commensurabilities following Mörth and Schlamminger (1979). The periods found in our analysis of the SSA components of polar motion (section 4) and of the derivatives of their envelopes (section 5) are labeled in red (there are 8, ranging from 1.2 to 165 years).

Note: Inspection of Table 1 may give the impression that there is a risk of "cherry picking". But certain periods that could have been reconstructed are not present, such as 103 yr that could have been obtained with Neptune. Commensurabilities are built from two consecutive planets and once their effect has been aggregated, they can be used in the next step of aggregation/commensurablity. The concept of commensurability is used by astronomers in order to discriminate between planets and other objects. The corresponding periods are not random: they are directly related to the revolutions of these bodies, and result from calculating means or subtracting periods two by two. Thus what can be obtained is not random. Moreover, as already pointed out by Mörth and Schlamminger (1979) or more recently Scafetta (2016), uncovering a limited number of common periods in a number of geophysical observables including sunspots cannot be due to chance. The action of kinetic moments of Jovian planets on the Sun's surface is what has allowed us to predict the next solar cycle from the ephemerids in a previous paper (Courtillot et al., 2021).

3-3 Singular spectral analysis: Finally we extract the relevant components of polar motion and ephemerids, and other long time series, with the help of *Singular spectral analysis* (SSA; Vautard and Ghil, 1989; Vautard et al, 1992; for more up to date versions of the technique by the St Petersburg school of mathematics, see Golyandina and Zhigljavsky, 2013). We have described and used our own version of SSA in a number of previous papers (e.g. Lopes et al., 2017; Le Mouël et al., 2021; Courtillot et al., 2021).

We discuss here a point that often comes up. An important factor in any time series analysis is the size of the window used in classical (Fourier) filters, to avoid erroneous interpretations (Kay and Marple, 1981). In SSA, the lagged-vector analysis window L should be sufficiently large so that each eigen vector carries a large part of the information contained in the original time series. In more mathematical words, one should work in the frame of Structural Total Least Squares (STLS) for a Hankel Matrix (Lemmerling and Van Huffel, 2001). A second issue is the separability of components. Many solutions are available, an exhaustive list being given by Golyandina and Zhigljavsky (2013, chap. 2.5.3, page 75). In this paper, we have used the *sequential SSA*. The window width L is variable, but remains close to 145 years.

4 – First Results

4.1 A striking reconstruction – Planets action on Earth's rotation: We start to test the ideas and use the tools summarized above by comparing the sum of the forces exerted by the four Jovian planets (using the IMCCE ephemerids) and the m_1 component of polar motion (1846 to 2020) as reconstructed from its SSA components, with the trend removed.

The polar coordinates m_1 and m_2 are related to the forces acting on Earth (Appendix 1). To first order, we can consider that the total "force" is simply proportional to the sum of individual (Jovian) planetary kinetic moments, plus the Solar kinetic moment. We have computed these moments from the planetary ephemerids, revolution periods and masses; their sum is given as the top black curve in Figure 3a. The red curve below is the reconstructed m_1 polar coordinate from Figure 1, after it has been decomposed in its SSA components, then reconstructed from them, but with the first component (the trend, see sub-section *4.2*) removed. Figure 3b shows an enlargement of the 1980-2019 part of Figure 3a. The correlation is quite striking. It is indeed expected, as already pointed out by Laplace (1799, book 5 in whole), that the Earth's rotation axis should undergo motions with components that carry the periods (and combinations of periods) of the Moon, Sun, and planets, particularly the Jovian planets as far as their kinetic moments are concerned (see also Mörth and Schlamminger, 1979, and Courtillot et al, 2021). This first exercise demonstrates that one should indeed consider planetary kinetic moments when describing the motions of the Earth's rotation axis. Based on this remarkable result, the aim of the rest of this paper is to see whether characteristic components of the ephemerids are also found in Earth's polar motion and other related (or not obviously related) phenomena.

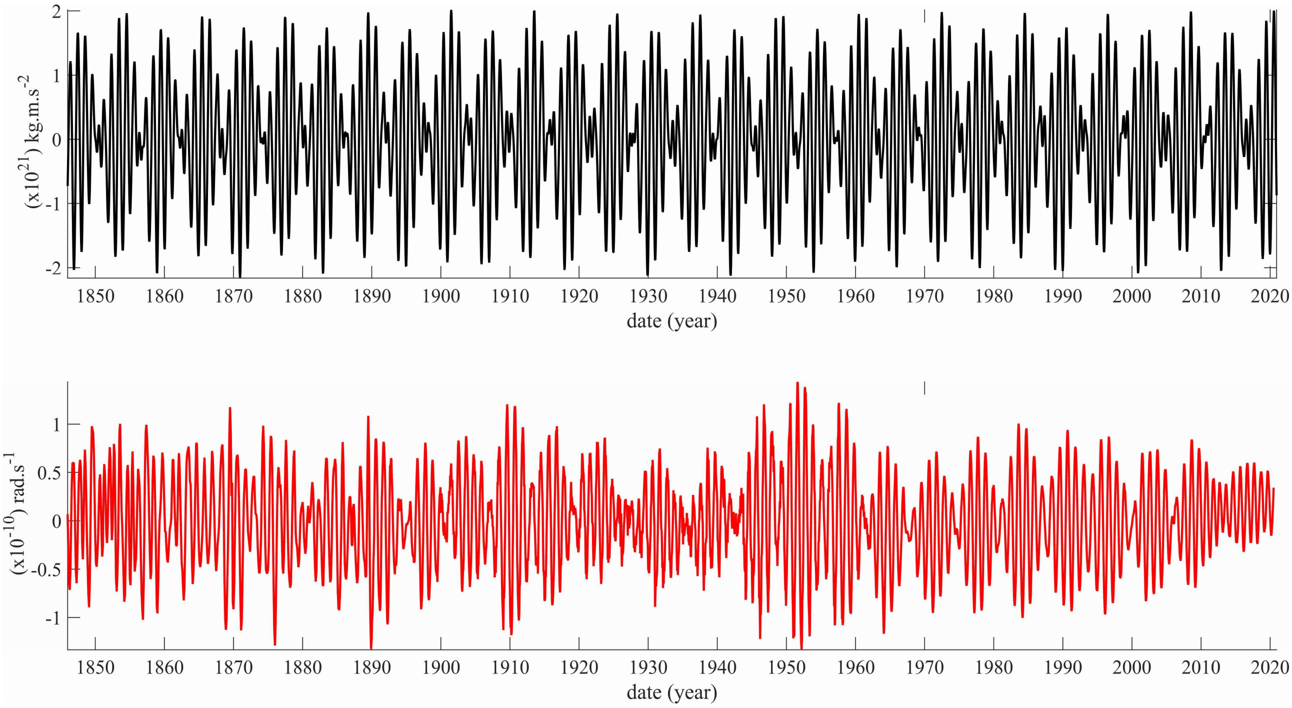

Figure 3a: Upper curve (in black) the sum of the forces of the four Jovian planets affecting Earth. Ephemerids from the IMCCE. Lower curve (in red) the m_1 component of polar motion (1846-2020) reconstructed with SSA and with the trend (Markowitz) removed.

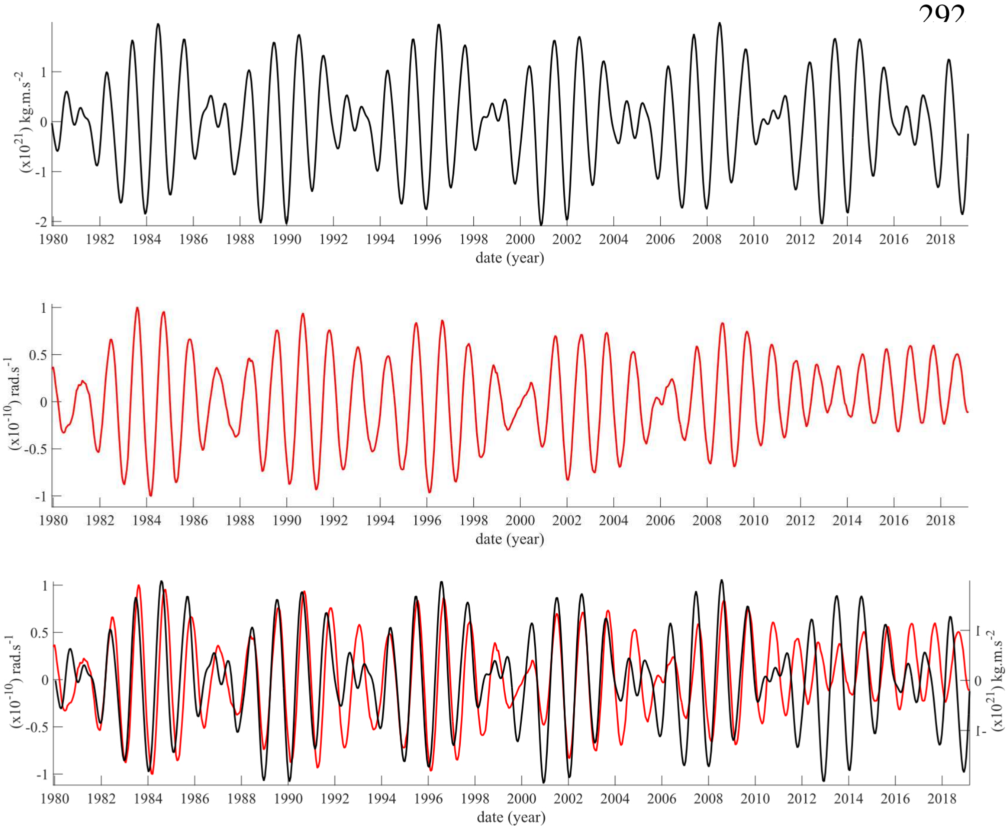

Figure 3b: Enlargement of Figure 3a (1980-2019) and superposition of the 2 curves.

	Planet		Commensurability (ε)	I/J	Associated periods (yr)
1	Mercury	Venus	-6.6×10^{-3}	2/5	0.18, 0.42
		Earth	10×10^{-3}	1/4	0.38, 0.62
		Mars	8.3×10^{-3}	1/8	0.78, 1.02
2	Venus	Earth	10×10^{-3}	3/5	0.19, 0.80
		Mars	5.6×10^{-3}	1/3	0.59, **1.20**
3	Jupiter	Saturn	1.9×10^{-3}	3/8	9.79, **21.64**
		Uranus	-1.7×10^{-3}	1/7	36.06, 47.91
4	Uranus	Neptune	9.6×10^{-3}	1/2	**40.40**, 124.37
		Pluto	6.0×10^{-3}	1/3	81.75, **165.72**
5	Pluto	Neptune	-8.3×10^{-4}	2/3	41.35, 206.13
	1	2	-6.4×10^{-3}	1/4	0.29, 0.49
			-1.3×10^{-3}	1/5	0.41, 0.60
			2.5×10^{-3}	1/2	0.01, 0.79
			6.5×10^{-3}	7/9	0.11, 0.91
	3	4	-7.7×10^{-3}	1/4	15.30, 25.09
			-3.2×10^{-3}	1/9	**57.29**, 67.08
			1.9×10^{-3}	5/9	9.38, 31.02
			6.5×10^{-3}	1/6	51.36, **73.00**
	3	5	1.3×10^{-3}	1/4	15.78, 25.57
			-6.3×10^{-3}	1/9	98.17, 107.96
			2.3×10^{-3}	5/2	9.85, **31.49**
			6.5×10^{-3}	1/6	**92.24**, 113.88

Table 1: Commensurable periods of pairs and pairs of pairs of planets computed following Mörth and Schlamminger (1979). The periods encountered in the present paper are printed in red.

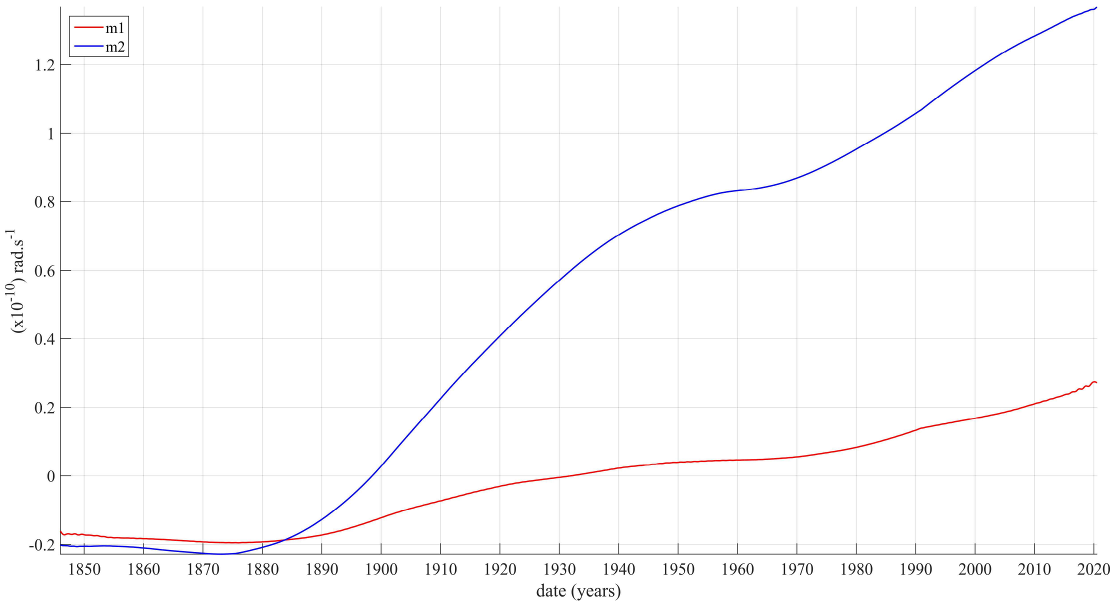

Figure 4: First SSA components of polar motion (trend) since 1846, called the Markowitz drift. Component m_1 is in red and m_2 in blue.

We next analyze one by one the leading SSA components of the Earth's rotation pole coordinates.

4.2 First SSA Component (Markowitz)

The first SSA components, shown in Figure 4, correspond to the mean trend of polar motion called the Markowitz drift (Markowitz, 1968). The drift velocity is on the order of 13 cm/yr and is principally carried by the E-W component m_2. As noted by a reviewer, these curves show changes in slope and inflection points that are reminiscent of the recent evolution of the Earth's global surface temperature (Le Mouël et al., 2019b, Figure 20). This important point is not discussed further in the present paper.

4.3 Second SSA Component (Annual)

The second SSA component is the forced annual oscillation (Figure 5). On that annual oscillation, Lambeck (2005, chapter 7, page 146) writes "*The seasonal oscillation in the wobble is the annual term which has generally been attributed to a geographical distribution of mass associated with meteorological causes. Jeffreys in 1916 first attempted a detailed quantitative evaluation of this excitation function by considering the contributions from atmospheric and oceanic motion, of precipitation, of vegetation, and of a polar ice. Jeffreys concluded that these factors explain the observed annual polar motion, a conclusion that is still valid today*".

Figure 5 shows that the annual components of m_1 and m_2 are significantly modulated, and in different ways (recall that the excitation functions are sums of sinuses and cosines with constant weights; Lambeck, 2005, page 153, equations 7.1.9). In the generally accepted theory, modulation is thought to be a response to reorganization of oceanic and atmospheric masses. We note in the modulation of m_1 the suggestion of a periodicity on the order of 150 years or more that could correspond to the Jose (1965) 171.5 yr cycle. Note that, given uncertainties, the Jose cycle could actually be the Suess- de Vries ~200 yr cycle (Stefani et al. 2020).

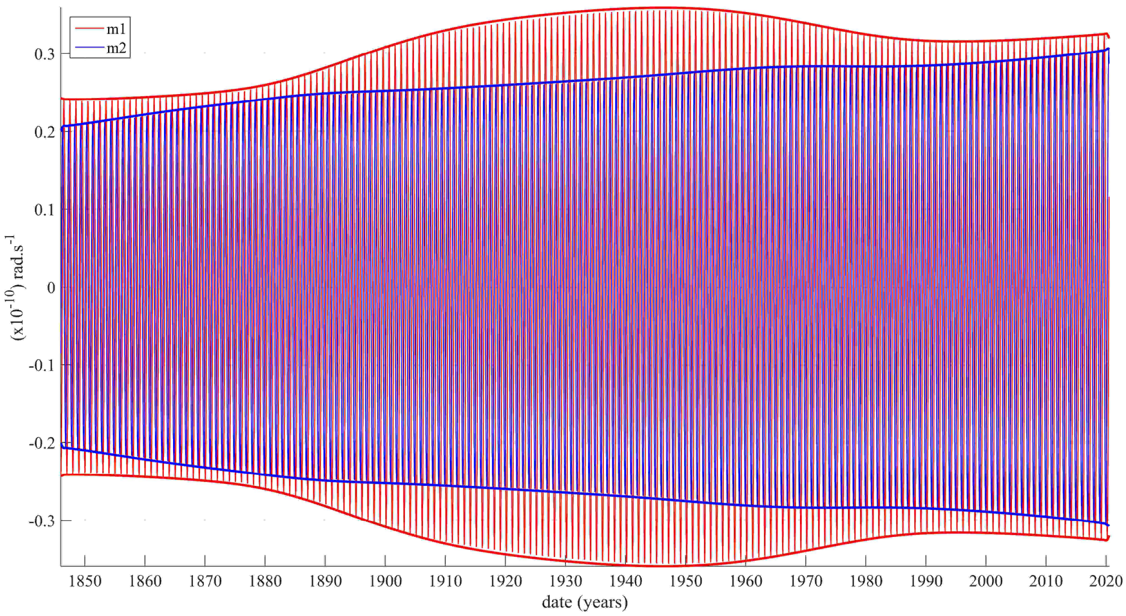

Figure 5: 2nd SSA comp. of polar motion (annual oscillation) since 1846. (m_1 red, m_2 blue).

4.4 Third SSA Component (Chandler)

Figure 6 shows the third SSA component, that is the Chandler component. Its amplitude is twice that of the annual component and its behavior is very different. The modulations are very large, similar for m_1 and m_2, and undergo a sharp and simultaneous change in phase and amplitude in 1930. Many scientists have studied this phase change (Hinderer et al., 1987; Runcorn et al., 1988; Gibert et al, 1998; Bellanger et al, 2001; Bellanger et al, 2002; Gibert and Le Mouël, 2008). The

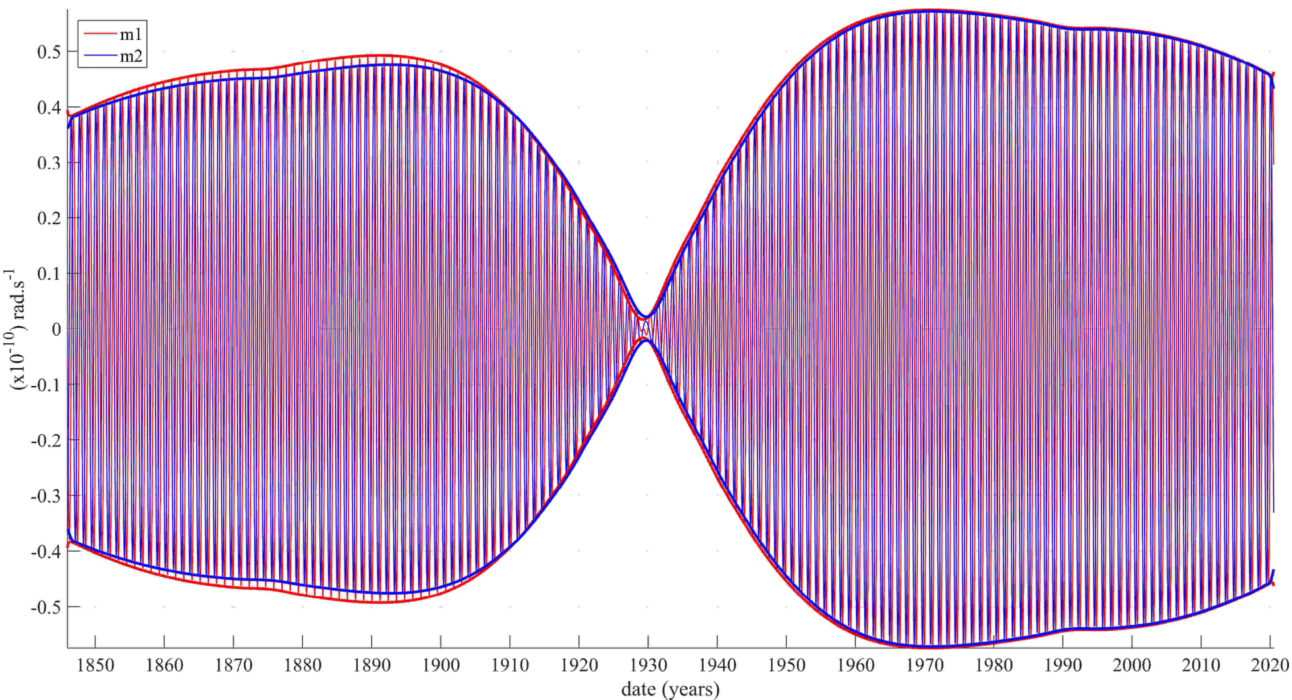

Figure 6: Third SSA components of polar motion (Chandler oscillation) since 1846. Component m_1 is in red and m_2 in blue.

Chandler oscillation extracted by SSA is similar to that obtained with wavelets by Gibert et al. (1998). It is also as regular as that obtained with SSA by Gorshkov et al. (2012).

When the first three SSA components of m_1 and m_2 are added, they account for 73% of the original variance. The quality of that incomplete reconstruction is shown in Figure 7.

Pushing the SSA analysis further reveals an oscillation with period 1.22 yr with an 18.6 yr modulation (the nutation), one with period 1.15 yr with a symmetrical modulation as in the case of the Chandler term, one with period 1.10 yr. Some of these (quasi-) periods have already been found using SSA on time series of sunspots (Le Mouël et al, 2020a). These periods seem to be linked to the ephemerids of solar system planets, which has been used by Courtillot et al (2021) to predict the

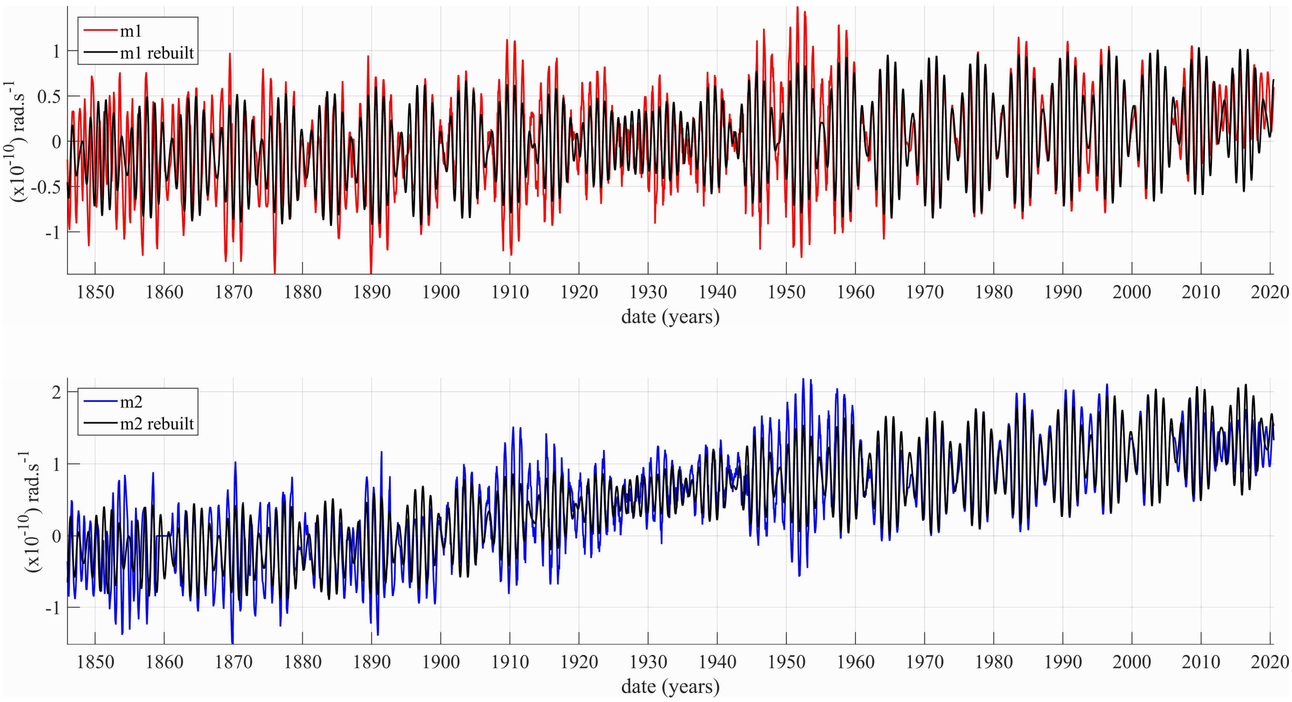

Figure 7: Reconstruction of polar motion since 1846 using only its first three SSA components. Top: observed component m_1 in black and reconstructed in red; bottom: observed component m_2 in black and reconstructed in blue.

date and amplitude of the next solar (Cycle 25) maximum. Other components are compatible with the Schwabe (11 yr) and Hale (22 yr) cycles. The 5.5 yr cycle is often associated with the Schwabe cycle (Usoskin, 2017), but not all authors agree. Moreover, these components are found only in m_2

and are much smaller in amplitude, on the order of 10^{-13} to 10^{-14} rad.s^{-1} vs 10^{-10} to 10^{-11} rad.s^{-1} for the first three (Lopes et al, 2017; Japaridze et al, 2020). When all components from the trend to the Hale cycle are added, they account for 95% of the total variance of the original series. Except for the 1.10 and 1.15 yr components, all others are found in the table of planetary interactions (Table 1).

5 - On Some Derivatives of SSA Components of Polar Motion

System (D) expresses that there is a link between a force and the derivative of the resulting polar motion (Appendix 1, equation 2). In other words Earth acts as a natural integrator (Appendix 1, equation 2 implies that m is an integral of ξ; see Le Mouël et al, 2010). This leads us to analyze the derivatives of the first three (largest) SSA components identified in the previous section.

5.1 Markowitz Drift

We first calculate the derivative of the Markowitz drift (Figure 4), and analyze its major SSA components. They are a trend (Figure 8a), a 90 yr pseudo-cycle (Figure 8b), a 40 yr pseudo-cycle (Figure 8c), a 22 yr period and an 11 yr period (Figure 8d).

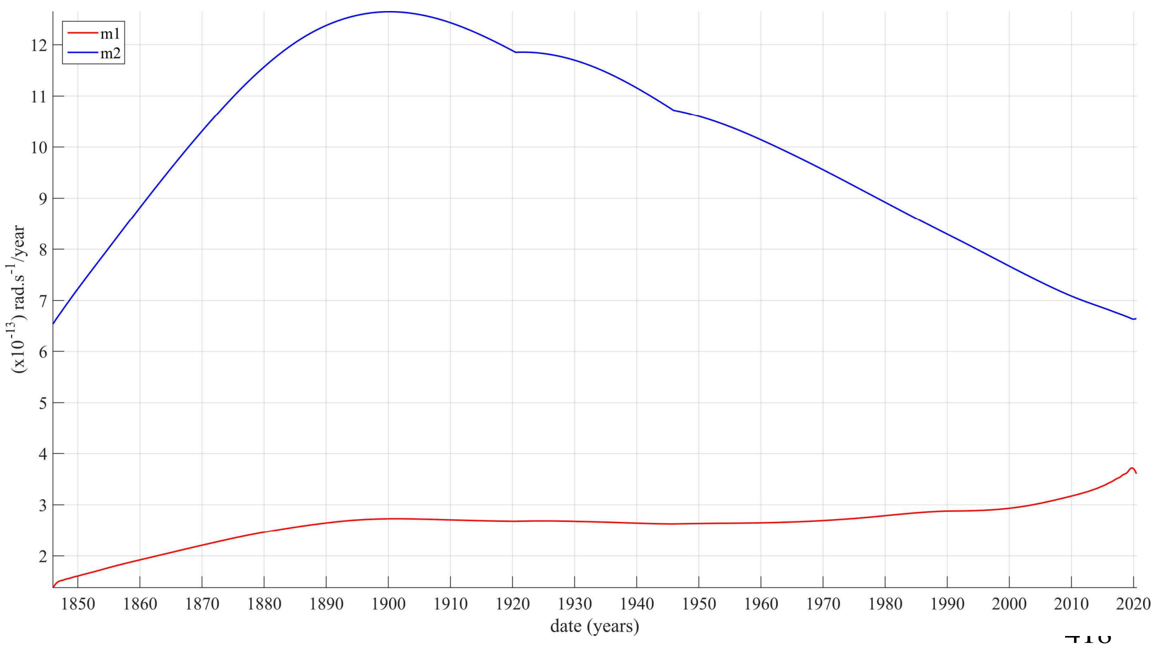

Figure 8a: First SSA component (trend) of the derivative of the Markowitz drift (first SSA component of polar motion). Component m_1 in red and m_2 in blue.

The 90 yr cycle is strongly correlated with the Gleissberg (1939) solar cycle (Figure 8b). Le

Mouël et al (2017) have obtained a period of 90 ± 3 yr from sunspot series SN_m_tot_V2.0[2]. It corresponds to a characteristic period in the ephemerids of Uranus (Table 1). There is also a close correspondence of periods for the 11 yr oscillation (Figure 8d). The drift could be linked to the modulation and varying "periodicity" of sunspots (8 to 13 yr). This is close to a characteristic period of Jupiter's ephemeris. The trend (Figure 8a) could be linked to the Jose (1965) 171.5 yr cycle, attributed to Neptune (Table 1) or to the Suess-de Vries ~200 yr cycle (Stefani et al., 2020). Finally, the 40 yr component has been shown by Mörth and Shlamminger (1979; see also Courtillot et al, 2021) to correspond to a commensurable revolution period of the four Jovian planets. It is interesting to point out that in both terrestrial polar motion and solar activity (as studied through the proxy of sunspots) the first 3 components that emerge from SSA are a trend, then the Gleissberg and Schwabe quasi-cycles.

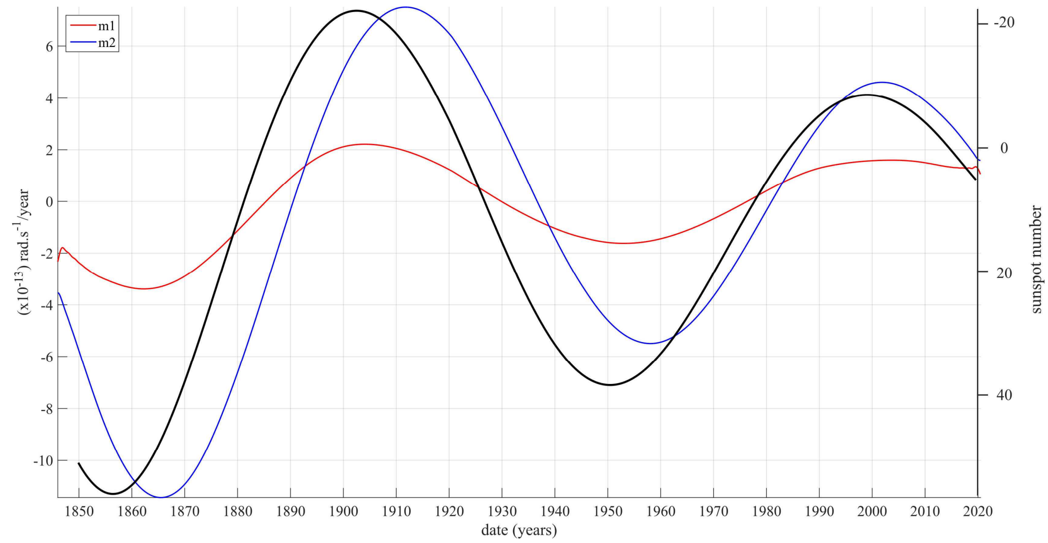

Figure 8b: Second SSA component (90 yr period)) of the derivative of the Markowitz drift (first SSA component of polar motion). Component m_1 in red and m_2 in blue. In black: Gleissberg cycle extracted from sunspots (sign reversed).

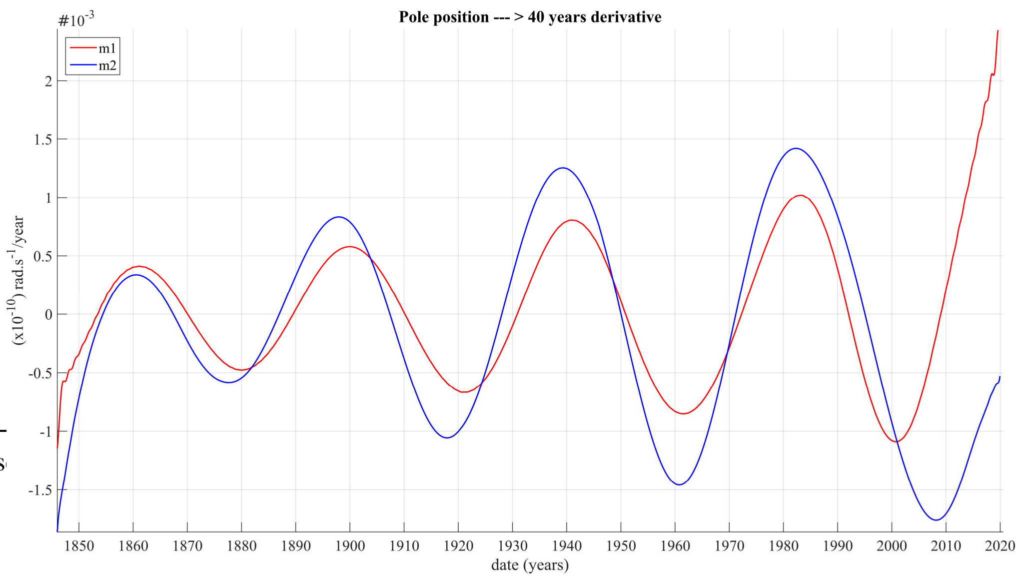

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Figure 8c: Third SSA component (40 yr period)) of the derivative of the Markowitz drift (first SSA component of polar motion). Component m_1 in red and m_2 in blue.

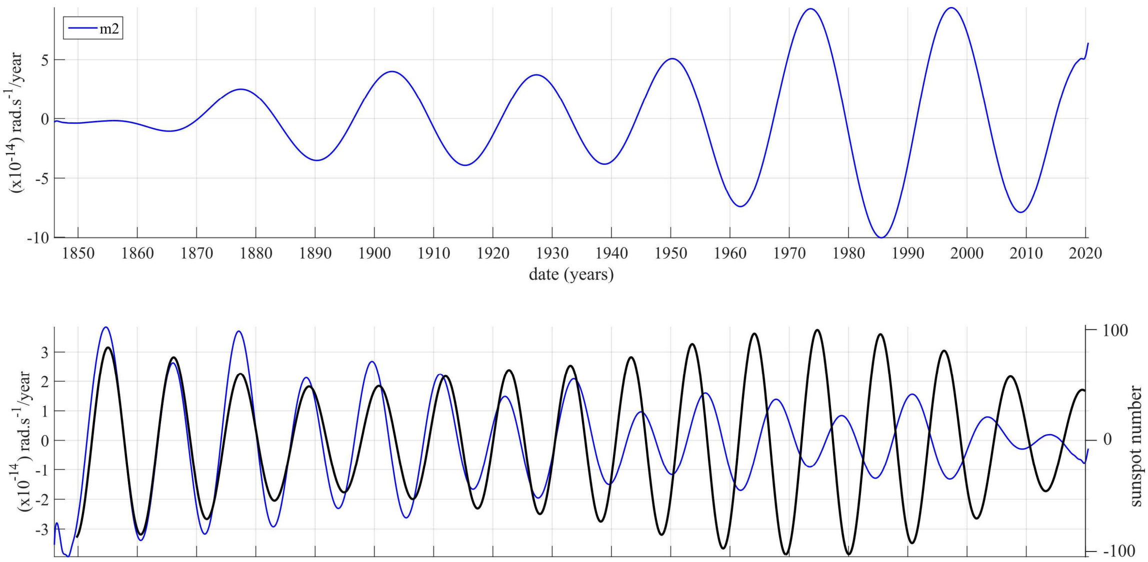

Figure 8d: 22 yr SSA component (top, component m_2 in blue) and 11 yr SSA component (bottom, component m_2 in blue) of the derivative of the Markowitz drift (first SSA component of polar motion). Bottom, black curve: the 11 yr Schwabe cycle extracted by SSA from the sunspot series (sign reversed).

5.3 Envelope of the Forced Annual Oscillation

We next turn to the derivative of the envelope of the forced annual oscillation (Figure 5). Its first SSA component, the trend, is shown in Figure 9a. For m_1 this trend is compatible with a little more than one period of a sine curve with a period close to 170 yr, that is the Jose (1965) solar cycle, corresponding to the ephemeris of Neptune (or again given uncertainties to the Suess-de Vries ~200 yr cycle). The next SSA component is a 70 yr cycle for m_1 and a 60 yr cycle for m_2 (Figure 9b). These periods, or pseudo-periods, are among those resulting from combinations of ephemerids of the Jovian planets (Mörth and Schlamminger, 1979; Scafetta, 2020; Table 1). The 60 yr cycle had already been found in sunspot series by Scafetta (2010) and Le Mouël et al (2020a). We had also seen it as an important component of series of global temperature and PDO and AMO oceanic indices (Courtillot et al, 2013).

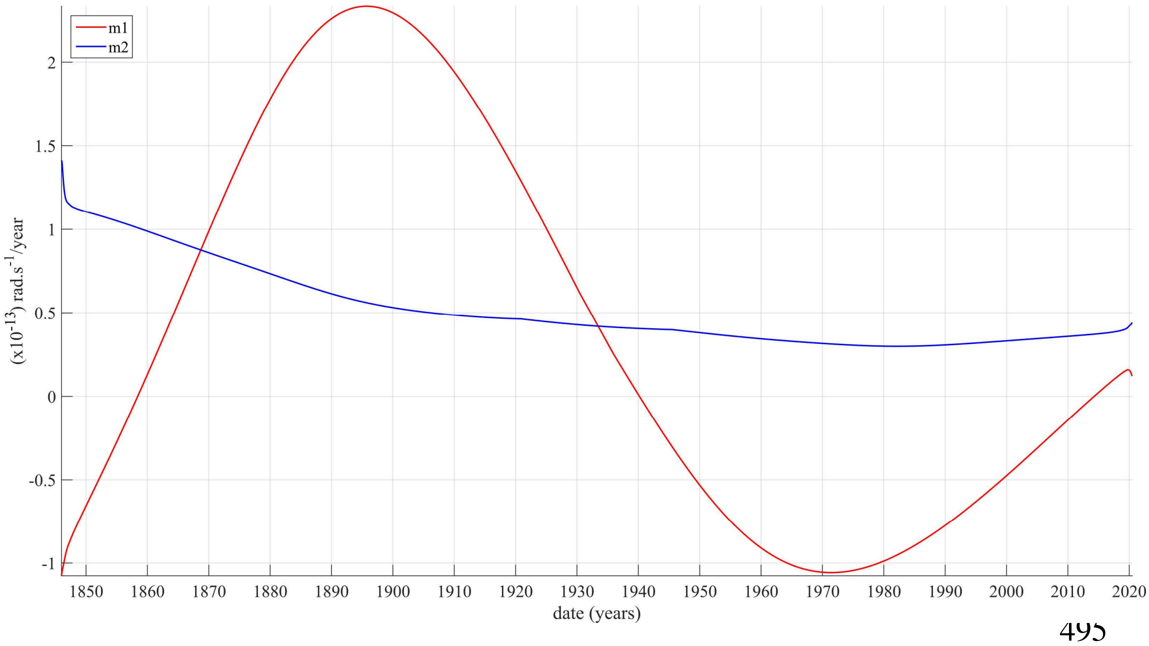

Figure 9a: First SSA component (trend) of the derivative of the envelope of SSA component 2 (annual oscillation) of polar motion.

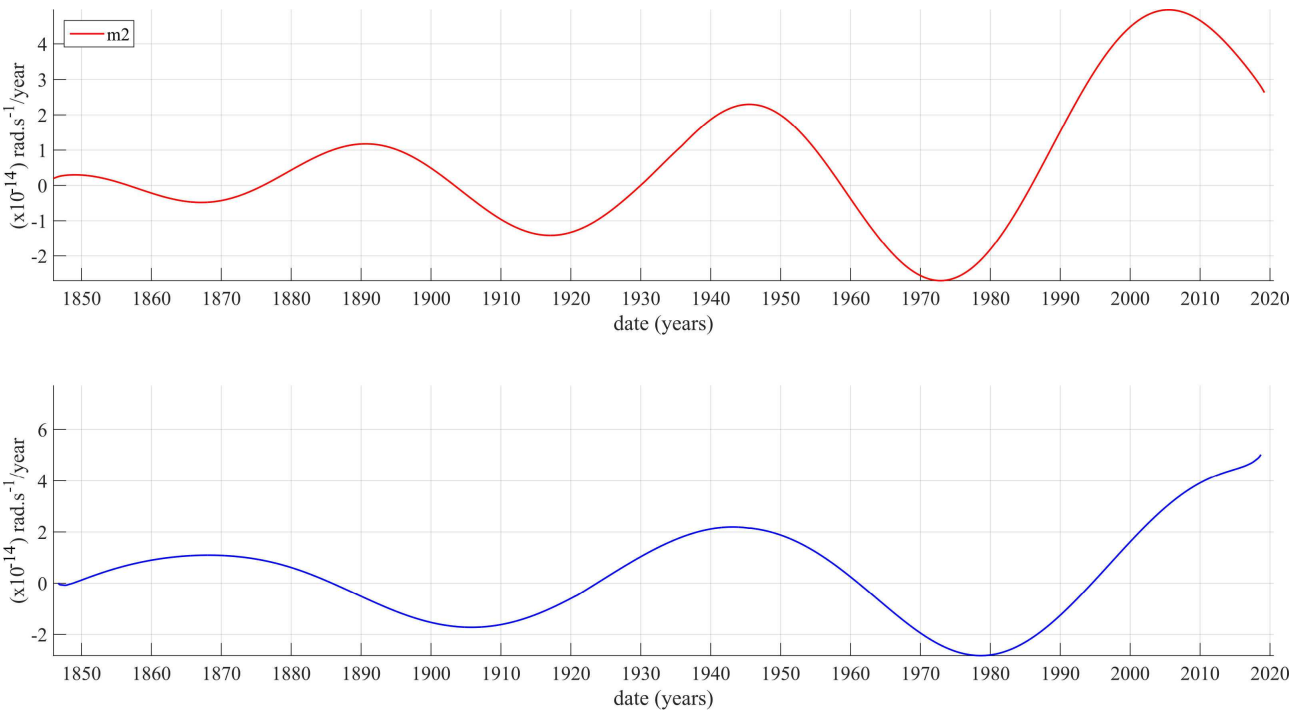

Figure 9b: Second SSA component (60 years and 70 years) of the derivative of the envelope of SSA component 2 (annual oscillation) of polar motion.

5.4 Free Chandler Oscillation

We now undertake the SSA analysis of the derivative of the envelope of the Chandler oscillation (Figure 6). We find components with periods 70, 40, 30 and 22 yr (Figures 10a to 10d). It is remarkable that the components for m_1 and m_2 are quasi-identical and have a very regular behavior, close to sine functions but with some slower modulation: they could be described as "astronomical" (as opposed to "astrophysical", as defined by Mayaud, 1980).

6 – Further Examples

We can illustrate further how Jovian planets influence polar motion with the combined effects of the pair Uranus (84 yr*)* – Neptune (165 yr): this pair has revolution periods compatible with the envelopes in Figures 5, 6 and 8b. Figure 11a shows the sum of the kinetic moments of these two Jovian planets as seen by Earth, which explains the annual oscillation whose envelope is of interest to us. Using again SSA, we have extracted the first four components of that signal; we show three

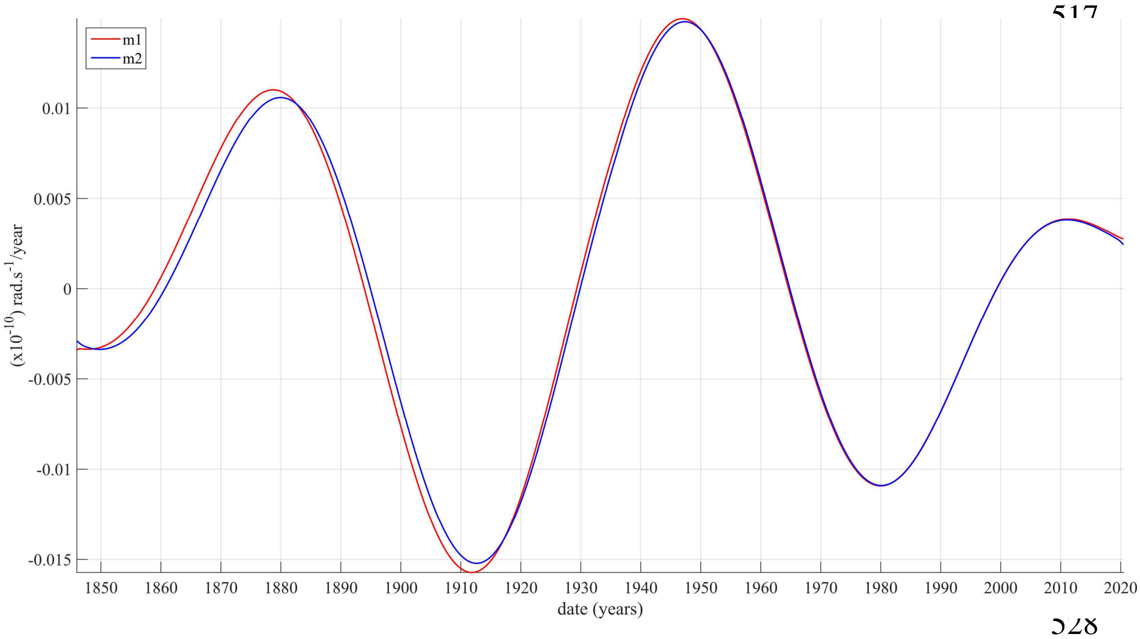

Figure 10a: First SSA component (70 yr quasi-period) of the derivative of the envelope of the Chandler oscillation.

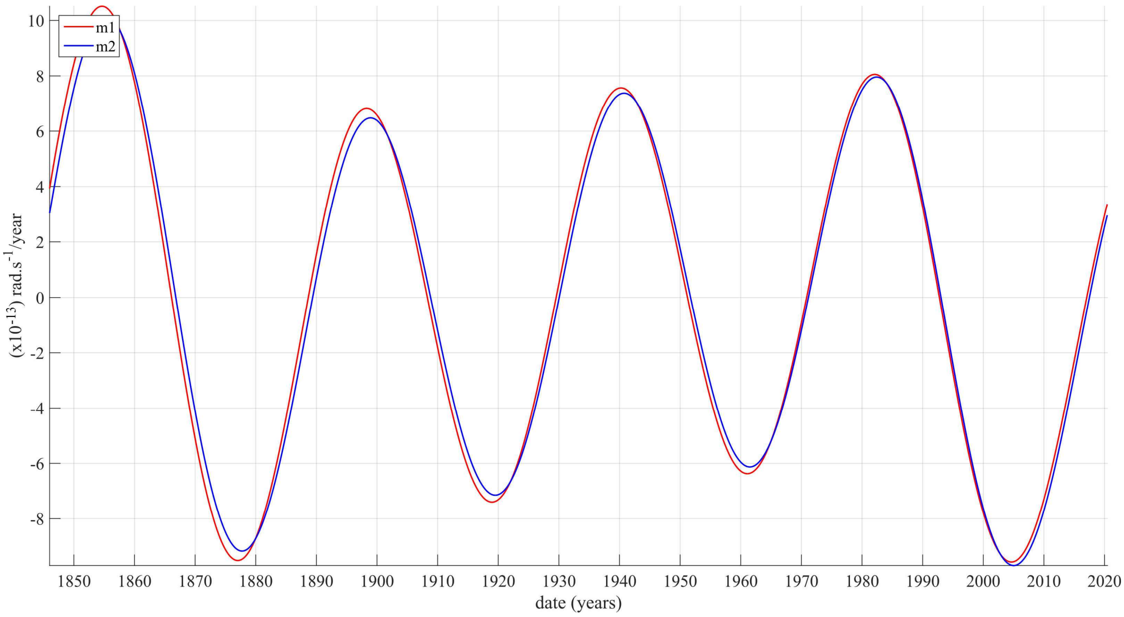

Figure 10b: Second SSA component (40 yr quasi-period) of the derivative of the envelope of the Chandler oscillation.

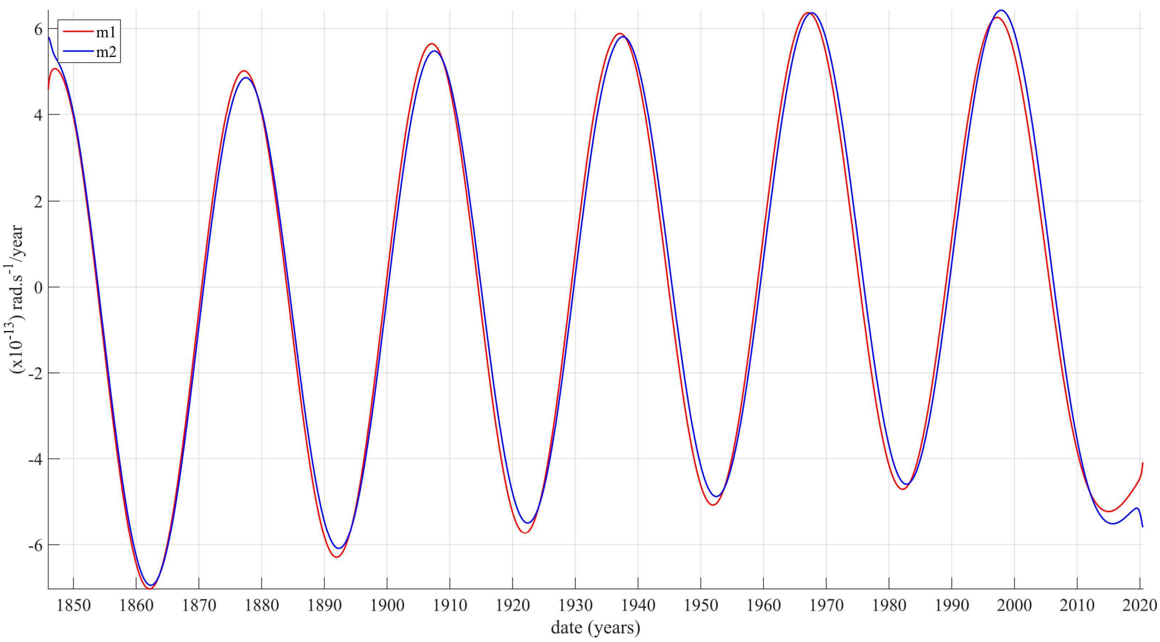

Figure 10c: Third SSA component (30 yr quasi-period) of the derivative of the envelope of the Chandler oscillation

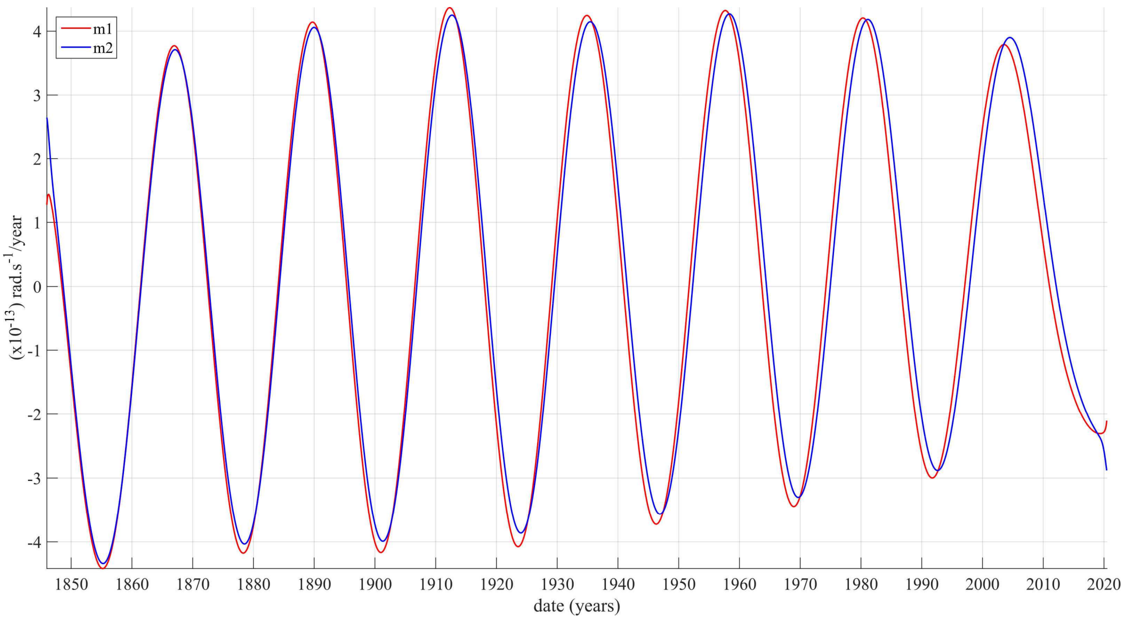

Figure 10d: Fourth SSA component (22 yr quasi-period) of the derivative of the envelope of the Chandler oscillation

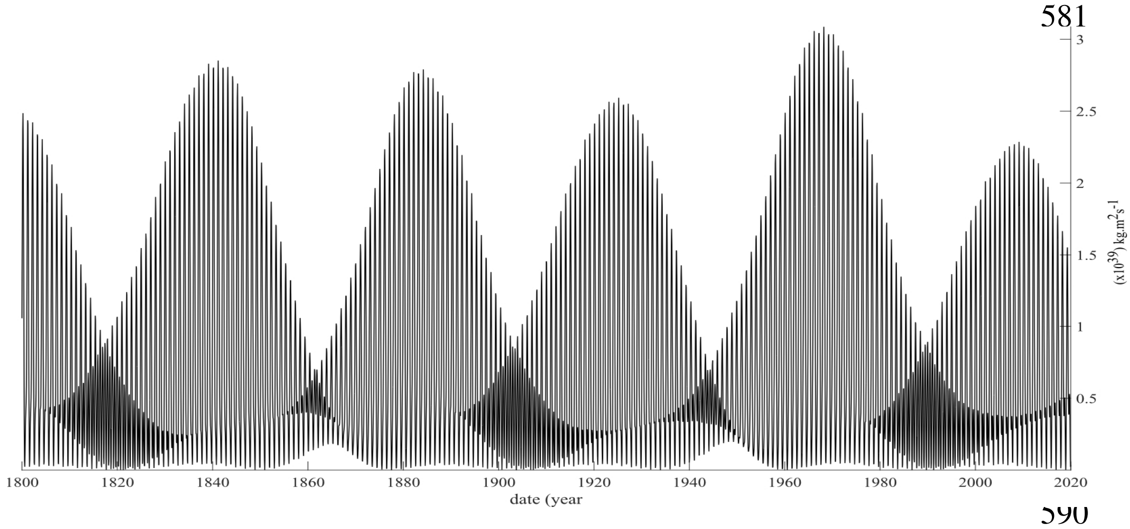

Figure 11a : The sum of the kinetic moments of Uranus and Neptune in a geocentric reference frame.

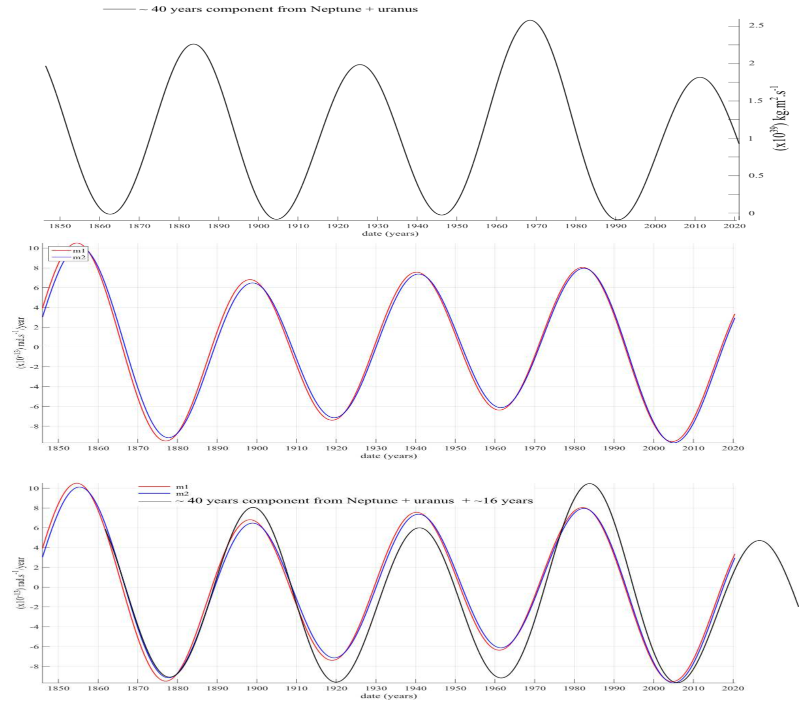

Figure 11b: Superposition (bottom) of the 40yr SSA (first) component of the curve shown in Figure 11a (top) and the component of polar motion with similar pseudo period in Figure 10b (middle). The top curve has been shifted by 16 yr with respect to the middle one in the bottom comparison.

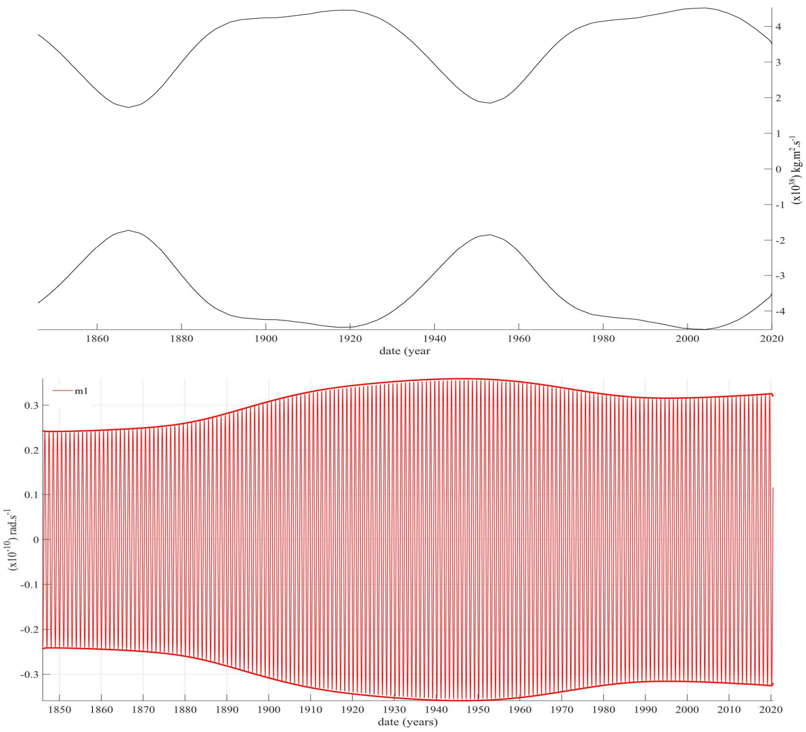

Figure 11c: Second component of the Uranus-Neptune pair (top) and forced annual oscillation of the polar motion m_1 (bottom).

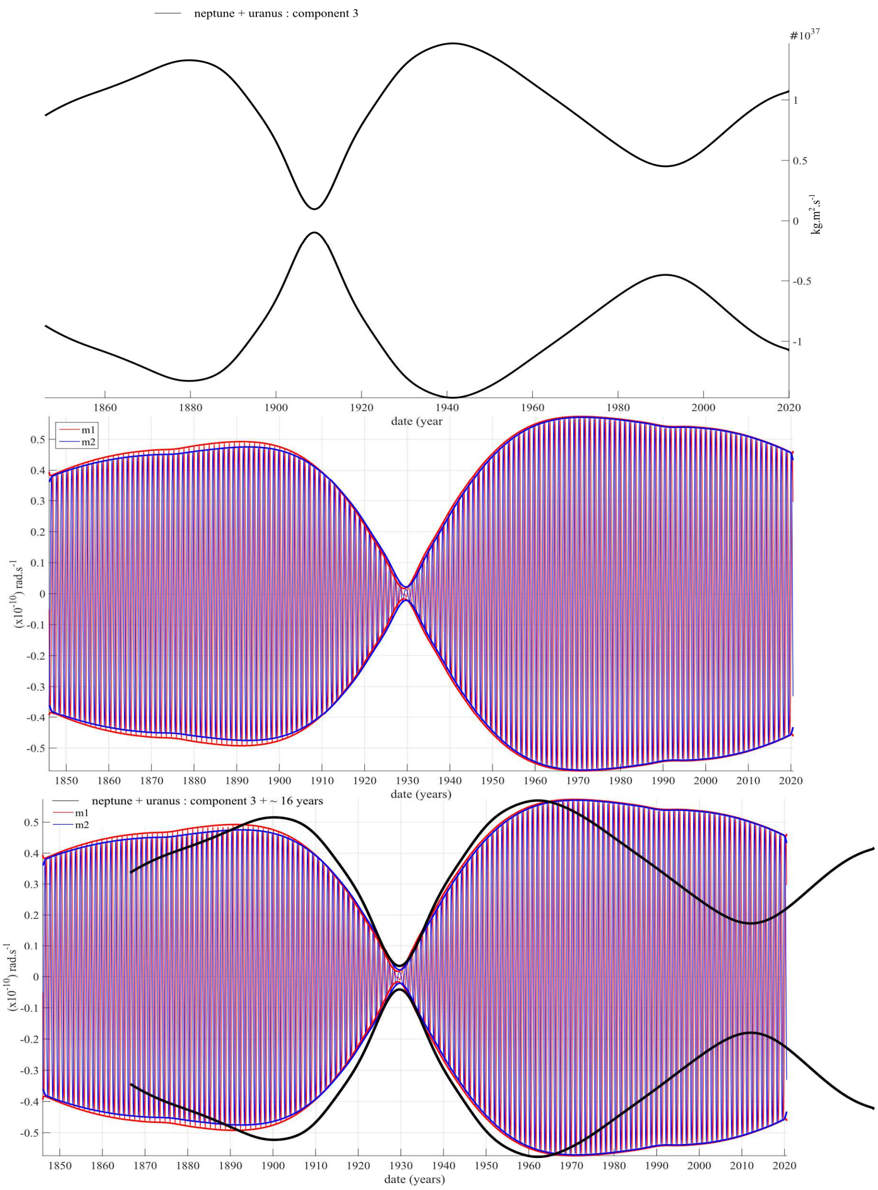

Figure 11d: Third SSA component of the Uranus-Neptune couple (top), Chandler oscillation (third SSA component of polar motion; middle) and comparison of the two with a 16 yr offset.

of them, from which the annual and semi-annual oscillations are absent. The first one (Figure 11b) has a pseudo period of about 40yr, similar to the envelope of the component of polar motion shown in Figure 10b. When a 16yr phase shift is taken into account, the correlation is remarkable (Fig 11b, bottom frame). The second component has nodes and bellies that are reminiscent of the envelope of the annual forced m_1 component of polar motion (Figure 11c). The third component of the Uranus-Neptune pair compares well with the Chandler free oscillation, particularly so when a 16 yr offset is taken into account (Figure 11d).

Finally, in Figures 12a to 12c, we superimpose the signatures (components) of the ephemeris of Jovian planets on the components of polar motion. In Figure 12a, the 90 yr component of the envelope of m_2 matches the ephemeris of Uranus offset by 32 years. In Figure 12b, the 165 yr component of the envelope of m_1 matches the ephemeris of Neptune, also offset by 32 years. In Figure 12c, the 30 yr component of the envelope of m_1 of the Chandler oscillation matches the ephemeris of Saturn offset by 15 years. The 11 yr component detected in the m_2 component of the derivative of the Markowitz drift (Figure 8d) has a variable phase drift with respect to the ephemeris. But, whereas "solar" components (periodicities) do appear at 22 and 11 yr (and 5.5 yr?) in polar motions, they are 3 to 4 orders of magnitude smaller than the leading components we discuss here.

We have seen that the sum of the Markowitz drift, annual oscillation and Chandler oscillation explain some 70% of polar motion. The same is true for the leading components of sunspots, i.e. the sum of the trend (Jose ~171.5 yr cycle), Schwabe cycle (~11 yr) and Gleissberg cycle (~90 yr) (on the same time range). These periods correspond to those of Neptune (~165 yr), Uranus (84 yr) and Jupiter (11.8 yr).

Many if not most of the (quasi-)periods found in the SSA components of polar motion, of their modulations, of their derivatives can be associated with the Jovian planets. Only one, the 432-434 day period is due to the Earth's mass and moments of inertia and not to the Jovian planets, as predicted by Laplace (1799).

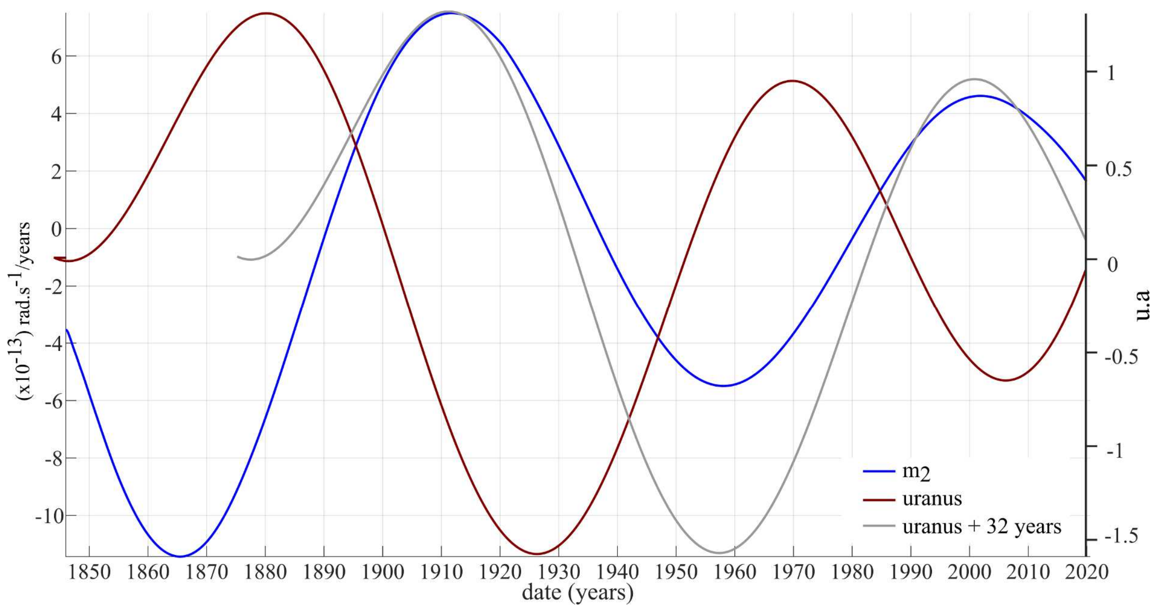

Figure 12a: Superimposition of the ~90 yr SSA component of the envelope of m_2 (blue curve; see

Figure 8b) with the ephemerids of Uranus (red curve). Grey curve: Uranus curve delayed by 32 years.

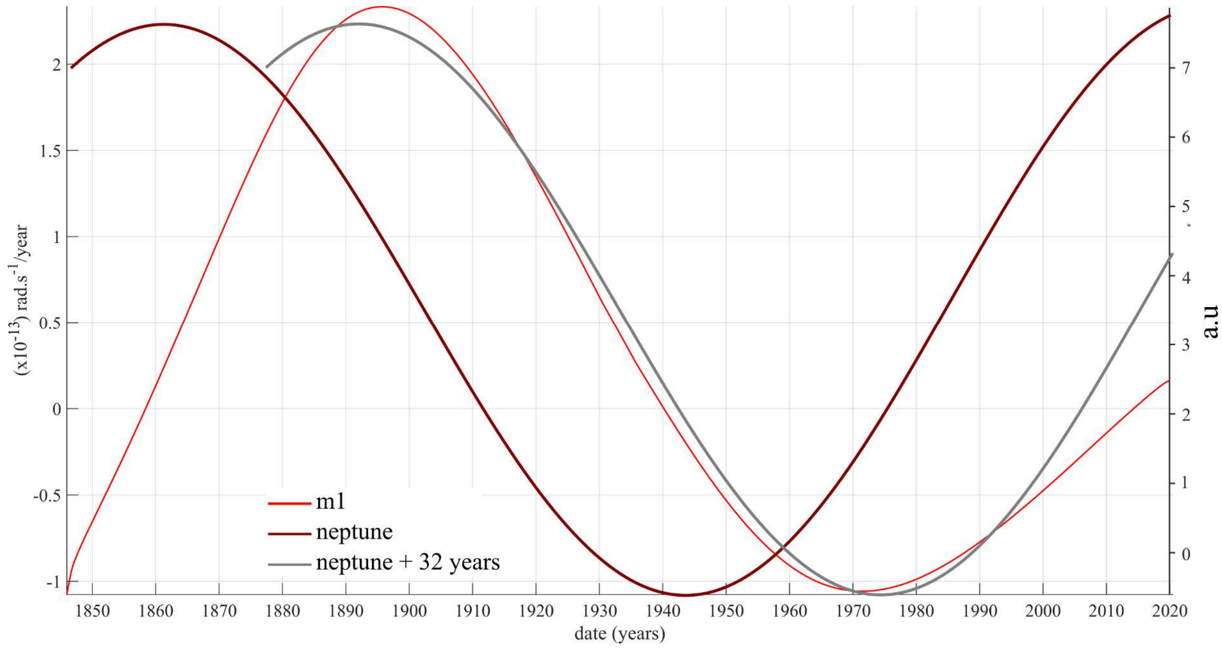

Figure 12b: Superimposition of the ~165 yr SSA component of the envelope of m_1 (red curve; see Figure 10a) with the ephemerids of Neptune (dark red curve). Grey curve: Neptune curve delayed by 32 years.

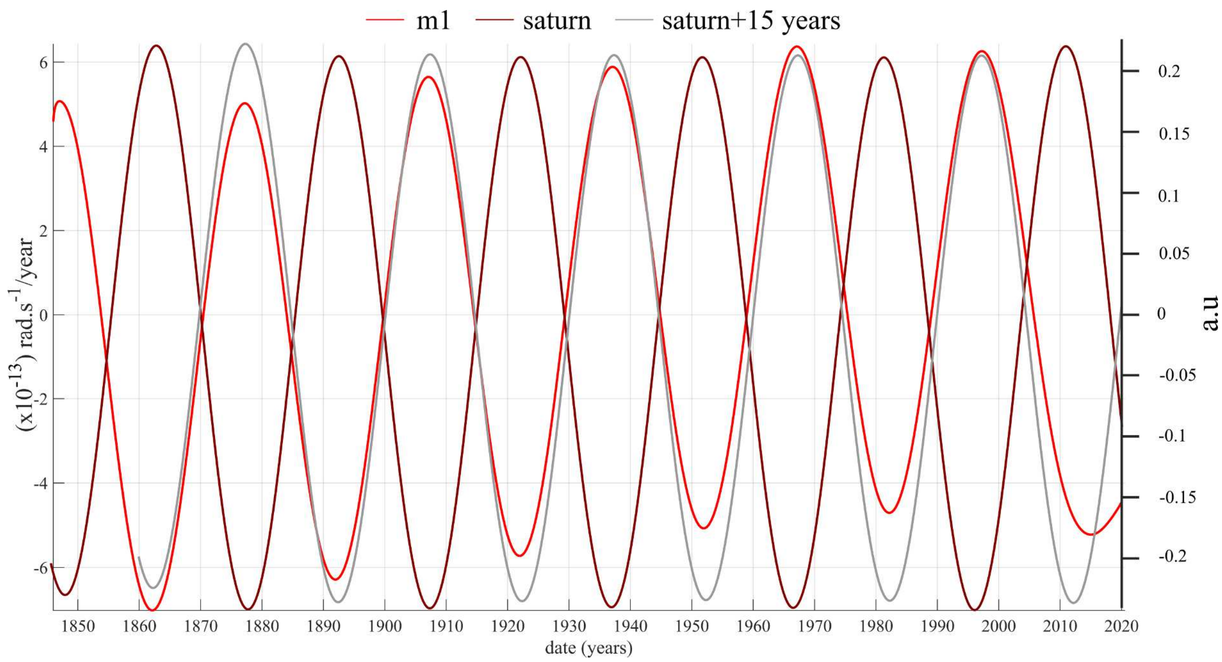

Figure 12c: Superimposition of the ~30 yr SSA component of the envelope of m_1 (red curve; see Figure 10c) with the ephemerids of Saturn (dark red curve). Grey curve: Saturn curve delayed by 15

years.

7 – Summary, Discussion and Conclusion

The general laws that govern the motions of celestial bodies have been derived and discussed by Laplace (1799) in his remarkable *Traité de Mécanique Céleste*. Laplace established the system of linear differential equations now known as the Liouville-Euler equations. He provided the full set of equations for the three Euler angles that specify the motions of a body's axis of rotation. Laplace differs from most later authors in the way he uses the Liouville-Euler system. Laplace makes full use of the system (D) for a rotating body that undergoes both rotation and translation, and solves the algebraic transcendant equations of Appendix 3, given all astronomical parameters. Most others use a simplified version with the formalism of excitation functions (Appendix 1, equation 2; a second order system) in which the possibility of a translation of the body's rotation axis is denied.

When Laplace obtains system (D) on page 74 of Chapter 7 of Book I, after 7 chapters that led him to these equations, he recognizes the fact that the system accounts for rotation as well as translation of a rotating body's polar axis. When Lambeck (for instance) follows the same route, his Chapter 3 (entitled «*Rotational Dynamic*») on page 30 begins with the following sentence : «*The fundamental equations governing the rotation of a body are Euler's dynamical equations*». Lambeck links the angular momentum to the torque that generated it. One means only rotation: that would be valid if the Earth's inclination were zero or a constant. The equations are the same, but one soon forgets that the momentum that is the source of the torque (Lambeck's system (3.1), page 30) is a 3D vector (with no reason to be restricted to 2D, since the Earth is neither flat, nor is its inclination constant; its rotation axis revolves about the Sun and is therefore subjected at least to our star's kinetic momentum). This oversight has some consequences. Since one only considers rotations, not translations, then the (Chandler) free rotation is obtained by zero-ing all torques and disregarding the third equation for the m_3 polar coordinate (that is assumed constant). Then, the forced annual oscillation cannot be due to the revolution of Earth about the Sun and one must find causes for these forced oscillations (the excitation functions). Laplace of course knew that polar coordinates m_1 and m_2 were connected to m_3. Therefore, Laplace did not constrain polar motion to the two surface components (m_1, m_2) but represented it by two meaningful components, the axis' inclination θ and the time derivative of its declination ψ depends on the inclination (previously calculated as a solution of the first Liouville-Euler equation). Laplace showed that there existed a free oscillation that would drift with a period between 306 (conjunction nodes) and 578 days (conjunction bellies), fully determined by the Earth's moments of inertia. This free oscillation, the Chandler oscillation, has a current value of 432-434 days. We now have long time series, up to a couple of centuries long, available and we use series of coordinates of the rotation pole m_1 and m_2 (Figure 1) to extend some of Laplace's (1799) results. A simple Fourier transform (Figure 2) shows the dominant spectral lines at 1 yr (forced annual oscillation) and 1.19 yr (free Chandler oscillation).

Singular spectral analysis (SSA) allows to better characterize the three leading components, the trend (~13cm/yr) called the (free) Markowitz drift (Figure 4), then the (forced) annual oscillation (showing different modulations for m_1 and m_2, Figure 5) and the Chandler oscillation (with a very large modulation and a phase change in 1930, similar for m_1 and m_2, Figure 6). Under the current theory, modulation is thought to be a response to reorganization of oceanic and atmospheric masses (e.g. Lambeck, 2005, chapter 7). Taken together, the first three SSA components explain 73% of the signal's total variance (Figure 7). The smaller components that follow have (pseudo-) periods of 1.22 (with an 18.6 yr modulation), 1.15 and 1.10 yr. Some of these periods have been encountered in sunspot series and in the ephemerids of Jovian planets (Le Mouël et al., 2020a; Courtillot et al., 2021).

We have next analyzed in the same way the envelopes of the derivatives of the first three SSA components of polar motion (Figure 8). We find a trend in the derivative of the Markowitz drift, that could also correspond to the 171.5 yr Jose cycle (associated with the ephemeris of Neptune) or the ~200 yr Suess-de Vries cycle). Next, a 90 yr component, reminiscent of the Gleissberg solar cycle (associated with the ephemeris of Uranus), a 40 yr component, corresponding to a commensurable revolution period of the four Jovian planets, a 22 yr and an 11 yr component, that can be associated with Jupiter and/or the Sun. For the modulation of the annual component of polar motion, SSA finds periods of 165, 70 and 60 years (Figure 9). The 60 yr component has been found in sunspots, global temperature of Earth's surface, and the oceanic oscillation patterns PDO and AMO (and Saturn). Finally, for the Chandler component, excellent matches are found for m_1 and m_2 with periods of 70, 40, 30 and 22 years (Figure 10).

One can think in Laplace's terms that the kinetic moments of planets act directly on Earth, or that these moments act on the external layers of the Sun (which is a fluid mass) and perturb its rotation, hence its revolution and kinetic moment $\boldsymbol{M}$ (Appendix 1), eventually affecting the Earth's axis of rotation: classical mechanics allows the hypothesis that planets influence the Earth's rotation axis (hence climate and other global phenomena). Laplace (1799) has shown that one should consider the orbital kinetic moments of all planets and that the Earth's rotation axis should undergo motions with the signatures/periods of the Sun and planets: the moments of the Jovian planets range from 1.7 to 19.3 10^{42} $kg.m^2.s^{-1}$, and for the Sun, an equivalent is 1.7 10^{41} $kg.m^2.s^{-1}$. To first order, the total kinetic moment applied to the Earth's rotation axis is simply the sum of individual (Jovian) planetary kinetic moments plus the Solar kinetic moment.

We have shown (Figure 3) that the m_1 component of polar motion reconstructed with SSA, with the Markowitz trend removed, matches remarkably well the sum of kinetic moments of the four Jovian planets. We have also computed these kinetic moments from the planetary ephemerids of Uranus and Neptune (Figure 11a). They "predict" remarkably well (Figure 11b) the 40 yr SSA component of the derivative of the envelope of the Chandler oscillation (Figure 11d).

We have previously determined the characteristic SSA components of solar activity, using sunspot numbers as a proxy (Le Mouël et al., 2019b). The sum of the Markowitz drift, annual oscillation and Chandler oscillation explain over 70% of polar motion. The same is true for sunspots, on the same time range, regarding the sum of the trend (Jose ~171.5 yr), Schwabe (~11 yr) and Gleissberg (~90 yr) cycles. These periods correspond to those of Neptune (~165 yr), Uranus (~90 yr) and Jupiter (~11 yr). We have superimposed the signatures (components) of the ephemerids of Jovian planets on the components of polar motion. The 90 yr component of the envelope of m_2 matches the ephemerids of Uranus, offset by 32 years (Figure 12a). The 165 yr component of the envelope of m_1 matches the ephemerids of Neptune, also offset by 32 years (Figure 12b). And the 30 yr component of the envelope of m_1 of the Chandler oscillation matches the ephemerids of Saturn, offset by 15 years (Figure 12c).

We have followed Mörth and Schlamminger (1979), who determined the commensurable periods of pairs and pairs of pairs of Jovian planets (Table 1): we find that 8 of them, ranging from 1.2 to 165 years, correspond to the SSA components of polar motion identified in sections 4 to 6 of this paper. Taken together, the curves shown in Figures 3 to 12 strongly argue that many components of Earth's polar motion can be explained by Laplace's classical celestial mechanics, with the combined instantaneous action of gravitational forces, and the longer, time-integrated action of kinetic moments. All celestial bodies act on the (m_1,m_2) pair of polar coordinates, a list that starts with the Sun and Moon, and continues with the Jovian planets. It would be satisfying to undertake a rigorous demonstration of the influence of all planets on the Sun and on the Earth's rotation axis, that is to calculate the inclination θ and declination ψ following Laplace's (1799) full treatment of the equations (Appendix 2). Since the Liouville-Euler equations are linear differential equations of first order, we have been able to use the frame of small perturbations and we have considered that the influence of planets can be taken as the sum of individual influences.

We have recalled that Lambeck (2005) writes: '*The seasonal oscillation in the wobble is the annual term which has generally been attributed to a geographical distribution of mass associated with meteorological causes […], a conclusion that is still valid today.'* When one works within this theoretical frame, there remain unexplained observations such as the 434 day value of the current period of the Chandler wobble or the 6 month component of oceanic indices (Le Mouël et al., 2019d). This has led to attempts to increase the complexity of the model, such as the forcing by climate or the visco-elastic response to glacial isostatic rebound. We have seen that this theory uses only 2 of the 3 Euler angles. By using the full system of equations in the Liouville-Euler system (D for Laplace), Laplace (1799) was able to go beyond the synthetic treatments of (for instance) Guinot (1977) or Lambeck (2005). We have seen in this paper numerous applications of this theory that explain many pseudo-periodic components of a number of geophysical (and solar) phenomena, making the leading role of planetary ephemeris clear.

The shorter periods (months to a few decades) often show as modulations of even shorter variations. And trends, with about 200 years of data, are possibly due to periods in the ephemeris comparable to or longer than the range of available observations. Still, these 200 years allow us to test Laplace's work further than he himself could. We have for instance been able to use this formalism to predict the future evolution of solar Cycle 25 (Courtillot et al, 2021).

It is widely assumed that both forced and free oscillations of Earth can, at least in part, be associated with climate forcings. Such has been the case from Jeffreys (1916) to Lambeck (2005), and recently to Zotov and Bizouard (2012) and Zotov et al. (2016). In all these works, causality is absent, be it from a time perspective or based on the orders of magnitude of the forces required to perturb the Earth's rotation. The periods that for instance Zotov et al. (2016) associate with an interaction between Earth's fluid and rigid envelopes are found in other geophysical phenomena such as the Earth's magnetic field or sunspots (Le Mouël et al., 2019a,b,c,d ; Le Mouël et al., 2020a,b ; Le Mouël et al., 2021; Courtillot et al., 2021 ; and references therein). We have come to the same conclusion regarding many climatic indices (Le Mouël et al., 2019d). If there is a good correlation of many characteristic periods, pseudo-periods and components extracted with SSA, for instance between Earth's rotation and many features of climate, it is reasonable to assume that this is because they are subject to some common forcings. This is not an overly speculative hypothesis: with the views of Laplace on tides, we know that the fluid envelopes react on short time scales (to changes in the Moon's declination for 2/3rds and the Sun for 1/3rd). On longer time scales, the whole (including solid) Earth responds (e.g. Dehant et Mathiews, 2015), all being governed by the Liouville-Euler equations.

In the present study, we have been able to find planetary signatures in polar motions, strictly based on observational data and using only classical mechanics. A possible causal chain thus emerges that has gravity potential and kinetic moments of planets acting directly or modulating motions of the fluid parts of celestial bodies, i.e. the Sun's outer layers (sunspots) and the Earth's atmosphere and ocean. These effects are in general not yet modeled: this is a domain where climate modeling warrants significant research advances.

In summary and conclusion of this work, two different mechanisms (causal chains) are likely at work. One is illustrated by the spectacular and direct effect of the kinetic moments of the (Jovian) planets on the Chandler wobble, whose intrinsic period (somewhere between 306 and 578 days) is synchronized to 433 days (a value that depends on Earth properties). The causal chain is directly from the Jovian Planets to Earth. Another causal chain would be an effect of planetary motions on the solar dynamo; variations in solar activity would in turn influence meteorological and climatic phenomena, such as mass transport between the equator and the poles, length of day, sea-level,... Given the remarkable coincidence between the quasi-periods of many of these phenomena, it is reasonable to assume that both causal chains are simultaneously at work. In that sense, it is not surprising to find the signatures of the Schwabe, Hale and Gleissberg cycles in many terrestrial phenomena, reflecting the characteristic periods of the combined motions of the Jovian planets.

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Appendix 1: Polar Coordinates and Excitation Functions

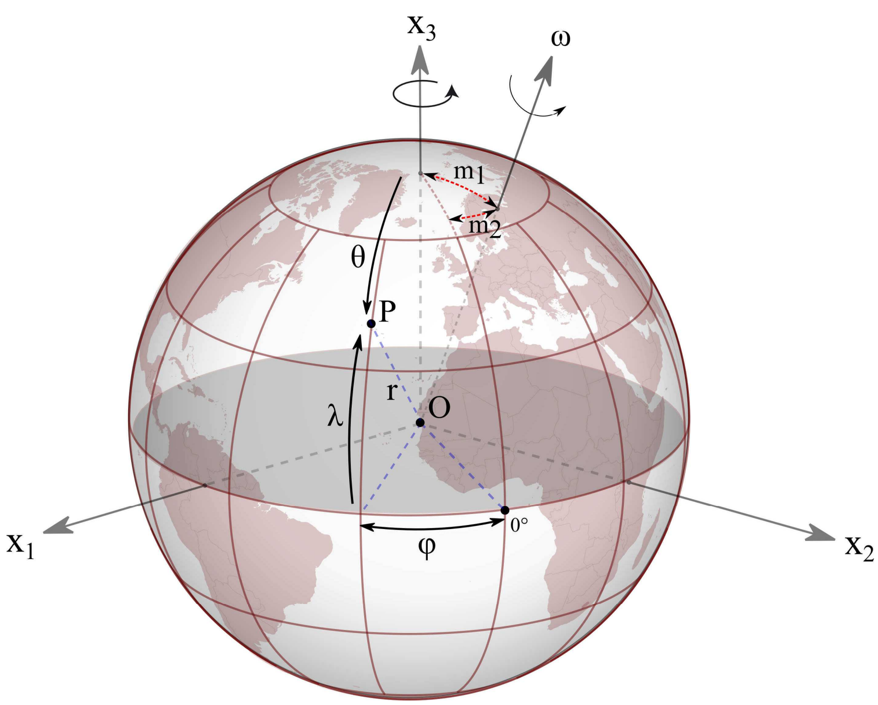

Figure A1 The reference system for the pole (m_1 and m_2).

Figure A1 gives the notations for the reference system that we use. The rotation pole is defined by its components m_1 and m_2, respectively on the Greenwich (0°) and 90°E meridians. We follow Lambeck's (2005, chapter 3) formalism. The rotation of the pole ω can be decomposed into three Euler angles (ω_1, ω_2, ω_3) associated with the axes (X_1, X_2, X_3) of the fixed reference frame. These Euler angles are a function of the Earth's mean angular velocity Ω and of the apparent position of the pole at the Earth's surface m_1, m_2, m_3 :

$$\omega_1 = \Omega m_1$$

$$\omega_2 = \Omega m_2$$

$$\omega_3 = \Omega(1 + m_3) \tag{1}$$

Noting that the Earth rotates about its axis and that its radius is constant, the Liouville-Euler system of equations (D for Laplace, 1799) becomes (1/σ is the Euler period):

$$j(\frac{\dot{\mathbf{m}}}{\sigma})\Omega + \mathbf{m} = \xi$$

$$\dot{m}_3 = \xi_3 \tag{2}$$

where **m** = $m_1 + jm_2$ and **ξ** = $\xi_1 + j\xi_2$. (ξ_1, ξ_2, ξ_3) are the excitation functions (forces and moments).

Appendix 2: Orientation of the Classical Orbit of a Planet Moving in the Field of a Central Body

The orientation of the classical orbit of a particle of mass μ (the Earth) around a large object of mass m (the Sun; m>>μ) is determined by two conservative vectors, the kinetic moment :

$$\vec{M} = \vec{r} \times \vec{p} \qquad \text{(B1)}$$

and the vector:

$$\vec{A} = \frac{\vec{p}}{\mu} \times \vec{M} - k\frac{m}{\mu}\frac{\vec{r}}{r} \qquad \text{(B2)}$$

whose conservation characterizes the Newtonian field (potential ϕ = - *km/r; k* being the constant of universal gravitation). Vector ***M*** is perpendicular to the orbital plane and vector ***A*** is oriented along the major semi-axis towards perihelion (Figure A2); the modulus of ***A*** is $km\mu e$, where e is the orbit's eccentricity. If m (the Sun) is rotating, then ***M*** is perturbed following the equation:

$$\frac{d\vec{M}}{dt} = \vec{\Omega} \times \vec{M}$$

Ω is the angular velocity of vector *A*, thus the revolution velocity of the perihelion.

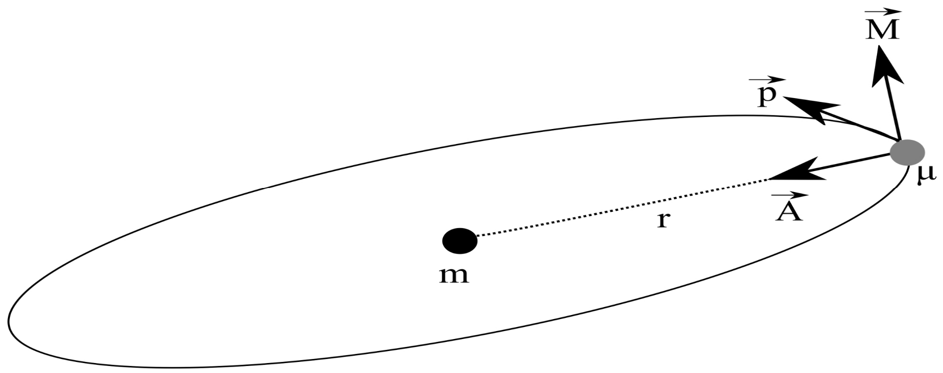

Figure B1: Particle μ (the Earth) rotates about the Sun (m>>μ). ***M*** is its kinetic moment and ***A*** derives from the Newtonian potential.

Appendix 3: Equations 5 and 6 from Laplace (1799), Book 5, Chapter 1, Pages 317-319

Laplace (1799) establishes the two equations that determine the joint influence of the Moon and the Sun on the inclination θ□of the rotation (polar) axis (Figure A1) as a function of time and the time derivative of its declination ψ, that is of its rotation (chapter 1 of book 5 of his treatise, in sections 5 and 6 to which the interested reader is invited to turn).

The first equation reads (in Laplace's terms):

$$\theta = h + \frac{3m}{4n}.(\frac{2C-A-B}{C}).\begin{cases} \frac{1}{2}.sin(\theta).\left\{cos(2\nu) + \frac{\lambda m}{m'}.cos(2\nu')\right\} \\ -(1+\lambda).m.cos(\theta).\sum.\frac{c}{f}.cos(ft+\varsigma) \\ +\frac{\lambda c'}{f'}.cos(\theta).cos(f't+\varsigma') \end{cases}$$

h corresponds to the inclination of the rotation axis, that was 26.0796° (Laplace, 1799, V (1), page 349), and is now 23.4333°. *m* is the Sun's mass and *m'* is the Moon's mass. *n* is the mean angular rotation velocity of Earth about its third principal axis, that is the length of day. *A*, *B*, *C* are the Earth's principal moments of inertia (*A*=*B*). Parameter λ is an integration constant obtained for the free oscillation of the Liouville-Euler system of partial differential equations, equal to:

$$n\sqrt{\frac{C-A}{C-B}AB} = \frac{C-A}{A} = \frac{\sigma}{\Omega}$$

$1/\sigma$ is the Euler period mentioned in the paper. ν is the Sun's and ν' the Moon's angular motion, that is their longitudes measured from the Spring mobile equinox or their right ascensions relative to Earth. One must also take into account the inclinations of Earth and Moon with respect to the Sun, represented by *c* and *c*'. They are associated to the longitudes of ascending nodes of Earth and Moon with respect to the Sun, also measured from the Spring mobile equinox and noted $(ft+\zeta)$ and $(f't+\zeta')$.

The second equation expresses the time variations of the rotation and uses the same notations and parameters:

$$\frac{d\psi}{dt} = \frac{3m}{4n}.(\frac{2C-A-B}{C}).\begin{cases} (1+\lambda).m.cos(\theta) - \frac{cos(\theta)}{2dt}.\left\{d.sin(2\nu) + \frac{\lambda m}{m'}.d.sin(2\nu')\right\} \\ +(1+\lambda).m.\frac{cos^2(\theta)-sin^2(\theta)}{sin(\theta)}.\sum.c.cos(ft+\varsigma) \\ +\lambda.m\frac{cos^2(\theta)-sin^2(\theta)}{sin(\theta)}.c'.cos(f't+\varsigma') \end{cases}$$

One sees the rotation as the sum of two oscillations, one intrinsic to the Earth linked to the constant $3m/4n$, that varies like $(1+\lambda).m.\cos\theta$ for all the nodes of luni-solar orbits, and is therefore a function of inclination θ, and another one forced by the Moon and Sun, linked to longitudes $(ft+\zeta)$ and $(f't+\zeta')$ and to right ascensions ν and ν' of these two orbs. Laplace therefore can estimate that the rotation period varies from 306 and 578 days.

Appendix 4: Original Text in French of the Quotation from Laplace

(*Traité de Mécanique Céleste*, vol. 5, cap. 1, page 347, 1799)

"Nous avons fait voir (n°8), que le moyen mouvement de rotation de la Terre est uniforme, dans la supposition que cette planète est entièrement solide, et l'on vient de voir que la fluidité de la mer et de l'atmosphère ne doit point altérer ce résultat. Les mouvements que la chaleur du Soleil excite dans l'atmosphère, et d'où naissent les vents alizés semblent devoir diminuer la rotation de la Terre: ces vents soufflent entre les tropiques, d'occident en orient, et leur action continuelle sur la mer, sur les continents et les montagnes qu'ils rencontrent, paraît devoir affaiblir insensiblement ce mouvement de rotation. Mais le principe de conservation des aires, nous montre que l'effet total de l'atmosphère sur ce mouvement doit être insensible; car la chaleur solaire dilatant également l'air dans tous les sens, elle ne doit point altérer la somme des aires décrites par les rayons vecteurs de chaque molécule de la Terre et de l'atmosphère, et multipliées respectivement par leur molécules correspondantes; ce qui exige que le mouvement de rotation ne soit point diminué. Nous sommes donc assurés qu'en même temps que les vents analysés diminuent ce mouvement, les autres mouvements de l'atmosphère qui ont lieu au-delà des tropiques, l'accélèrent de la même quantité. On peut appliquer le même raisonnement aux tremblements de Terre, et en général, à tous ce qui peut agiter la Terre dans son intérieur et à sa surface. Le déplacement de ces parties peut seul altérer ce mouvement; si, par exemple un corps placé au pole, était transporté à l'équateur ; la somme des aires devant toujours rester la même, le mouvement de la Terre en serait un peu diminué; mais pour que cela fut sensible, il faudrait supposer de grands changements dans la constitution de la Terre".

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