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Geophys. J. Int. (2005) **161,** 591–602 doi: 10.1111/j.1365-246X.2005.02612.x

GJI Geomagnetism, rock magnetism and palaeomagnetism

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G. Hulot[1,2] and C. Bouligand[1]

[1]*Département de Géomagnétisme et Paléomagnétisme, CNRS UMR 7577, Institut de Physique du Globe de Paris, 4 Place Jussieu,* 75252 *Paris cedex 05, France. E-mail: gh@ipgp.jussieu.fr*
[2]*NASA/GSFC, code 921, Greenbelt, MD* 20771, *USA*

Accepted 2005 February 9. Received 2004 December 21; in original form 2004 March 2

SUMMARY

In the present paper, we address symmetry issues in the context of the so-called giant gaussian process (GGP) modelling approach, currently used to statistically analyse the present and past magnetic field of the Earth at times of stable polarity. We first recall the principle of GGP modelling, and for the first time derive the complete and exact constraints a GGP model should satisfy if it is to satisfy statistical spherical, axisymmetrical or equatorially symmetric properties. We note that as often correctly claimed by the authors, many simplifying assumptions used so far to ease the GGP modelling amount to make symmetry assumptions, but not always exactly so, because previous studies did not recognize that symmetry assumptions do not systematically require a lack of cross-correlations between Gauss coefficients. We further note that GGP models obtained so far for the field over the past 5 Myr clearly reveal some spherical symmetry breaking properties in both the mean and the fluctuating field (as defined by the covariance matrix of the model) and some equatorial symmetry breaking properties in the mean field. Non-zonal terms found in the mean field of some models and mismatches between variances defining the fluctuating field (in models however not defined in a consistent way) would further suggest that axial symmetry also is broken. The meaning of this is discussed. Spherical symmetry breaking trivially testifies for the influence of the rotation of the Earth on the geodynamo (a long-recognized fact). Axial symmetry breaking, if confirmed, could hardly be attributed to anything else but some influence of the core–mantle boundary (CMB) conditions on the geodynamo (also a well-known fact). By contrast, equatorial symmetry breaking (in particular the persistence of an axial mean quadrupole) may not trivially be considered as evidence of some influence of CMB conditions. To establish this, one would need to better investigate whether or not this axial quadrupole has systematically reversed its polarity with the axial dipole in the past and whether dynamo simulations run under equatorial symmetric CMB conditions display additional transitions (mirror transitions, which we describe) only allowed in such instances. This remains to be fully investigated.

Key words: dynamo theory, geomagnetism, geostatistics, palaeomagnetism, spherical harmonics, statistical methods.

1 INTRODUCTION

Thanks to the availability of many historical observations of the magnetic field of the Earth over the past few centuries and of many more high-precision satellite measurements of the near Earth's magnetic field in the past few decades, excellent spherical harmonic (SH) models of the main field (MF) produced in the core of the Earth have been constructed in the recent years (e.g. Jackson *et al.* 2000; Langlais *et al.* 2003), making it possible to infer some aspects of the current dynamical behaviour of both the field itself and of the liquid core (see e.g. Hulot *et al.* 2002). However, the historical period only covers a very short period of time. To infer and understand the nature of the MF before that, we are forced to rely on indirect measurements available through human artefacts, lava flows and sediments that have been magnetized in the ancient field. Those magnetized samples can also be used to construct time varying SH models of the ancient field. However, this requires that enough samples are available at a given epoch and that they can be synchronized to well within the timescales over which the MF is likely to evolve. Because of those limitations, only the largest scales of the field over the past few millennia have yet successfully been modelled in this way (Hongre *et al.* 1998; Korte & Constable 2003).

A different approach is therefore required to access the information contained in the palaeomagnetic databases that are otherwise available for recent geological epochs (past 5 Myr). Two databases are of particular interest, the so-called palaeosecular variation (PSV) database (Quidelleur *et al.* 1994; Johnson & Constable 1996; McElhinny & McFadden 1997) and palaeointensity database (Tanaka *et al.* 1995). Both of them consist of data recovered from volcanic samples that have acquired their magnetization within much less time than needed for the MF to evolve significantly. Each sample in those databases can then be considered as an instantaneous spot value of the direction (for the PSV database) or of the full vector (for the palaeointensity database) of the ancient field, at a given known location, but at a relatively poorly known time. This time is nevertheless known with enough accuracy to identify the period of fixed polarity during which the sample acquired its magnetization. All samples corresponding to such a given period can therefore be expected to contain some statistical information about the local and global MF behaviour during that period of time.

Many different approaches have been used in the past to carry such analysis and derive so-called time-averaged field (TAF) and PSV models of the palaeomagnetic field (for a review, see e.g. Merrill *et al.* 1996). One approach in particular has proved successful in the recent years. The giant gaussian process (GGP) approach first introduced by Constable & Parker (1988) in a relatively restrictive form, next generalized by Kono & Tanaka (1995), and by Hulot & Le Mouël (1994) and Khokhlov *et al.* (2001) to also account for possible temporal and spatial correlations. This approach is particularly attractive as its formalism can be used to simultaneously analyse the historical (Constable & Parker 1988; Hulot & Le Mouël 1994), the archeomagnetic (Hongre *et al.* 1998) and the palaeomagnetic MF (Constable & Parker 1988 and many studies since, see e.g. Kono *et al.* 2000, Khokhlov *et al.* 2001 and references therein). None of the approaches proposed so far, including the more recent approach proposed by Love & Constable (2003), has so many powerful properties.

Like all previous approaches, the GGP approach however relies on a set of statistical assumptions, the most fundamental of which are required for the approach to be valid. Additional simplifying assumptions have also been introduced to ease the data analysis and reduce the number of free parameters needed to characterize the field behaviour. Although all those additional assumptions seem natural, they imply some restrictions on the way the field is *a priori* assumed to behave. It is the purpose of the present paper and of a companion paper (Bouligand *et al.* 2005, – this issue, hereafter referred to as Paper II) to investigate the meaning and validity of all those assumptions. This is done in two steps. First, by relying on symmetry considerations (present paper). We introduce the GGP approach and review the simplifying assumptions used so far. We next give the first derivation of the complete set of constraints a GGP model should satisfy if it is to satisfy spherical, axisymmetric or equatorial symmetric properties. This reveals that most of the assumptions used so far amount to make symmetry assumptions, albeit not always in a fully consistent way. We discuss the meaning of this and explain how symmetry properties could be used to better characterize the regime under which the geodynamo has been operating in the past, and possibly identify some influence of non-symmetrical core–mantle boundary (CMB) conditions. In a second step (Paper II), the issue is addressed from a more general point of view, by taking advantage of results from numerical simulations for which, contrary to the real-Earth case, all assumptions involved in GGP modelling can be tested.

2 GENERALIZED GIANT GAUSSIAN PROCESSES

At any given time t and location $\mathbf{r}$ outside the core, the MF is assumed to be curl-free, and is therefore written in the form

$$\mathbf{B}(\mathbf{r}, t) = -\boldsymbol{\nabla} V(\mathbf{r}, t), \tag{1}$$

where:

$$V(\mathbf{r}, t) = a \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^{n+1} \sum_{m=0}^{n} \left[g_n^m(t) Y_n^{mc}(\theta, \varphi) + h_n^m(t) Y_n^{ms}(\theta, \varphi)\right] \tag{2}$$

and a is an arbitrary reference radius, taken to be the planetary (Earth) radius where the field is usually observed; (r, θ, φ) are the spherical coordinates, $Y_n^{m(c,s)}(\theta, \varphi)$ the real SH functions of degree n and m, Schmidt quasi-normalized (this normalization is defined in more details in Appendix A) and the $[g_n^m(t), h_n^m(t)]$ are the so-called Gauss coefficients, which define a vector $\mathbf{x}(t)$ in a multidimensional model space.

Following and describing the evolution of the field in terms of a generalized GGP then amounts to assume that at times of stable polarity, $\mathbf{x}(t)$ can be described in terms of a single realization of a multidimensional stationary random Gaussian process in this model space. The field and its associated GGP model are then characterized by (like any multidimensional stationary random Gaussian process, see e.g. Gardner 1990):

a statistical mean (or mean model) $E\{\mathbf{x}(t)\} = \boldsymbol{\mu}$, (3)

a covariance matrix $E\{[\mathbf{x}(t) - \boldsymbol{\mu}][\mathbf{x}(t') - \boldsymbol{\mu}]^T\} = \boldsymbol{\gamma}(t' - t)$, (4)

where $E\{\}$ is the statistical expectation, $\boldsymbol{\mu}$ is a vector of components $[\mu(g_n^m), \mu(h_n^m)]$ defining a mean model in the model space, and $\boldsymbol{\gamma}(t' - t)$ is a matrix of elements $\gamma(x_n^m, x'^{m'}_{n'}, t' - t)$, with x and x' being either g or h.

A very useful property of such a stationary Gaussian process is that, provided its covariance matrix $\boldsymbol{\gamma}(\tau)$ decreases fast enough towards zero when τ becomes large, it is ergodic for both its mean and its covariance (Gardner 1990). In other words,

$$\lim_{T\to\infty} \bar{\mathbf{x}}_T = \boldsymbol{\mu} \quad \text{and} \quad \lim_{T\to\infty} \bar{\mathbf{K}}_T(\tau) = \boldsymbol{\gamma}(\tau),$$

where we have introduced

$$\bar{\mathbf{x}}_T = \frac{1}{T}\int_0^T \mathbf{x}(t)\,\mathrm{d}t, \tag{5}$$

$$\bar{\mathbf{K}}_T(\tau) = \frac{1}{T}\int_0^T [\mathbf{x}(t) - \bar{\mathbf{x}}_T][\mathbf{x}(t+\tau) - \bar{\mathbf{x}}_T]^T\,\mathrm{d}t. \tag{6}$$

Thus, if we further assume that the field only has a short-term memory, which we will indeed assume for reasons outlined later on, both eqs (5) and (6) can be used to estimate $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}(\tau)$ from temporal averages $\bar{\mathbf{x}}_T$ and $\bar{\mathbf{K}}_T(\tau)$ over the trajectory of $\mathbf{x}(t)$ in the model space.

In the GGP formalism and because of eq. (5), $\boldsymbol{\mu}$ may then be viewed as the TAF to be expected. In the same way and because of eq. (6), $\boldsymbol{\gamma}(\tau)$ may be viewed as defining the statistical properties of the fluctuating field about $\boldsymbol{\mu}$, hence of the PSV.

3 LOCAL EXPRESSION OF GGPS

When considering palaeomagnetic data, relying on eqs (5) and (6) to infer $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}(\tau)$ is not possible because we do not have a direct access to the Gauss coefficients [and thus to $\mathbf{x}(t)$]. We only have measurements of some field components at a single location $\mathbf{r}_i$ at a time t_i. Those provide some information about $\mathbf{x}(t)$ but only through eqs (1) and (2). However, the statistics produced by a GGP model for the field at any such location can also be predicted, and therefore used to recover some information about $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}(\tau)$.

This is straightforward for full vector measurements of the local field $\mathbf{B}(\mathbf{r}, t)$. The way a geomagnetic field model $\mathbf{x}(t)$ predicts $\mathbf{B}(\mathbf{r}, t)$ through eqs (1) and (2) can indeed always formally be written as

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{P}(\mathbf{r})\mathbf{x}(t), \tag{7}$$

where $\mathbf{P}(\mathbf{r})$ is a matrix. Because of this linear relationship, if the MF is assumed to be the result of a generalized GGP, $\mathbf{B}(\mathbf{r}, t)$ is predicted to behave as a realization of a 3-D stationary Gaussian process, the characteristics of which are directly related to those of the generalized GGP model through

$$E\{\mathbf{B}(\mathbf{r}, t)\} = \boldsymbol{\mu}_B(\mathbf{r}) = \mathbf{P}(\mathbf{r})\boldsymbol{\mu}, \tag{8}$$

$$E\left\{[\mathbf{B}(\mathbf{r}, t) - \boldsymbol{\mu}_B(\mathbf{r})][\mathbf{B}(\mathbf{r}, t') - \boldsymbol{\mu}_B(\mathbf{r})]^T\right\} \\ = \boldsymbol{\gamma}_B(\mathbf{r}, t' - t) = \mathbf{P}(\mathbf{r})\boldsymbol{\gamma}(t' - t)\mathbf{P}(\mathbf{r})^T. \tag{9}$$

Because of the short-term memory assumption, it also follows that this 3-D stationary Gaussian process is ergodic. Thus, in principle, temporal averages (of the type in eqs 5 and 6) can also be used to locally infer the values of $\boldsymbol{\mu}_B\,(\mathbf{r})$ and $\boldsymbol{\gamma}_B\,(\mathbf{r}, \tau)$. Those can then be used to try and invert eqs (8) and (9) for $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}(t' - t)$.

Very often, however, the palaeomagnetic data to be used are not $\mathbf{B}(\mathbf{r}, t)$, but only its direction. This can make the problem trickier, because those quantities are not linearly related to $\mathbf{x}(t)$. However, it turns out that the statistics predicted for such directional data by a given generalized GGP can also exactly be computed (Khokhlov *et al.* 2001).

Thus, in principle, a procedure can always be devised to try and seek which, if any, generalized GGP can account for any given set of data. However, because (i) not so much data are available, (ii) the number of parameters needed to characterize a generalized GGP [i.e. needed to define $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}(t' - t)$] is potentially very large, and (iii) it is not easy to invert the data for $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}(t' - t)$, simplifying assumptions have always been introduced in all GGP models.

4 ASSUMPTIONS AND RESULTS FROM PUBLISHED GGP MODELS

In their analysis of the historical and archeomagnetic data, Hulot & Le Mouël (1994) and Hongre *et al.* (1998) assumed that $\boldsymbol{\mu}$ reduces to an axial dipole, that all Gauss coefficients behave independently from one another and that all those sharing the same degree n also share the same autocovariance function, i.e. that the covariance matrix takes the diagonal form

$$\gamma\left(x_n^m, x_{n'}^{\prime m'}, \tau\right) = 0 \text{ if } x_n^m \neq x_{n'}^{\prime m'} \qquad \text{and} \quad \gamma\left(x_n^m, x_n^m, \tau\right) = \gamma_n(\tau), \tag{10}$$

where

$$\gamma_n(\tau) = \sigma_n^2 \exp\left[-\frac{\tau^2}{2\tau_n^2}\right] \tag{11}$$

and τ_n can then be viewed as the typical correlation time, associated to all Gauss coefficients of degree n and characterizing how fast $\gamma_n(\tau)$ [and thus $\boldsymbol{\gamma}(\tau)$] decreases to zero when τ increases. These assumptions made it possible to combine temporal with statistical averages, estimate the σ_n and infer values for the τ_n of the order of a few centuries (for the lowest degrees) down to a few decades (for degree 13).

Those results and assumptions were consistent with the original assumptions Constable & Parker (1988) had introduced even earlier to analyse the PSV database of the past 5 Myr. They had assumed that each sample would correspond to statistically independent realizations of a GGP, that $\boldsymbol{\mu}$ reduces to an axial dipole with a small axial quadrupole (which changes signs together with the axial dipole at times of reversals, an important assumption as we shall later see) and that the covariance matrix takes the diagonal form (10), but with

$$\gamma_n(\tau) = \sigma_n^2 \delta(\tau) \tag{12}$$

in place of eq. (11), where $\delta(\tau)$ is the Dirac function and σ_n^2 is assumed to be of the form $\sigma_n^2 = \sigma_n^2\,(CP) = \sigma^2 n[(n+1)(2n+1)]^{-1}$ for $n \geq 2$, defining a flat spatial spectrum for the non-dipole field at the CMB.

Subsequent studies however revealed that the PSV data required a more complex GGP model. Kono & Tanaka (1995) were for instance led to assume that the covariance matrix can take the more general form

$$\gamma\left(x_n^m, x_{n'}^{\prime m'}, \tau\right) = 0 \text{ if } x_n^m \neq x_{n'}^{\prime m'} \qquad \text{and} \quad \gamma\left(x_n^m, x_n^m, \tau\right) = \sigma_n^{m^2}\delta(\tau), \tag{13}$$

which amounts to assume that $\boldsymbol{\gamma}(\tau)$ is still forced to be diagonal, with negligible memory, but that it can now be a function of both degree n and order m. Kono & Tanaka (1995) indeed pointed out that the data could be explained by enhancing order 1 covariances and in particular σ_2^1, a suggestion soon confirmed by Quidelleur & Courtillot (1996).

As another interesting possibility and following an earlier suggestion of McFadden *et al.* (1988) that the dipole family component of the field [defined by all Gauss coefficients with $(n - m)$ odd] could behave independently from its quadrupole family component [Gauss coefficients with $(n - m)$ even], Kono & Tanaka (1995) also envisioned the possibility that the data could be explained by a GGP model satisfying eq. (13), but with $(\sigma_n^m)^2 = d\sigma_n^2\,(CP)$ if $(n - m)$ is odd and $(\sigma_n^m)^2 = (1 - d)\sigma_n^2\,(CP)$ if $(n - m)$ is even, where d is thus a parameter defining the relative contributions of each family. Hulot & Gallet (1996) considered this proposal in some detail and showed that this would require a value of $d \approx 0.9$ [independently of the exact value chosen for $\mu\,(g_1^0)$]. Most recently, Tauxe & Kent (2004) indeed confirmed that a model based on those assumptions (with $d = 0.93$) could be considered. However, Hulot & Gallet (1996) noted that this would imply that on average over the past 5 Myr, more than 90 per cent of the energy of the non-dipole field would have been concentrated in its dipole family component, in stark contrast with the present-day situation, which sees a historical field with equal energy in both families.

Bearing this drawback in mind and having further shown that it is definitely not possible to explain the PSV with a covariance matrix of the form of eqs (10) and (12), whatever the value of σ_n, Hulot & Gallet (1996) next suggested that a covariance matrix of an even more general form than eq. (13), involving cross-covariance terms, and different values for $\gamma(g_n^m, g_n^m, \tau)$ and $\gamma(h_n^m, h_n^m, \tau)$, also be considered. Constable & Johnson (1999) positively tested this last possibility against the PSV data.

The study of Hulot & Gallet (1996) also underlined the need to simultaneously address the determination of $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}(\tau)$ when dealing with PSV data. This need was not considered significant in the early studies, because it was felt that the data did not call for a more complex mean field than the axial (dipole plus small quadrupole) field assumed by Constable & Parker (1988). However, a series of investigations initiated by Gubbins & Kelly (1993), soon followed by others (see e.g. Johnson & Constable 1997, and references therein) suggested that non-zonal terms could be present in the TAF $\boldsymbol{\mu}$. Those conclusions have been questioned (McElhinny *et al.* 1996; Carlut & Courtillot 1998). Also, the method used to infer $\boldsymbol{\mu}$ in those studies has some drawbacks. It relies on means of the local field direction and assumes that those mean directions are only a function of $\boldsymbol{\mu}$ [and not of $\boldsymbol{\gamma}(\tau)$], which is incorrect (Khokhlov *et al.* 2001; see also the perturbation study of Hatakeyama & Kono 2001). In any case, all those results clearly show that neither $\boldsymbol{\gamma}(\tau)$ nor $\boldsymbol{\mu}$ should *a priori* be considered too simple and that both quantities should be inverted simultaneously when dealing with PSV data. This is not an easy task (see e.g. Hatakeyama & Kono 2002, for a recent attempt).

The task is a little easier, at least in principle, if one deals with palaeointensity data of the type assembled by Tanaka *et al.* (1995), also covering the past few Myr. This database is made of full vector field measurements, and an approach based on eqs (8) and (9) can be used to independently invert the data for both $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}(\tau)$. Kono *et al.* (2000) recently attempted such a computation. For that purpose, they assumed a covariance matrix of yet an even more general form:

$$\gamma\left(x_n^m, x_{n'}^{\prime m'}, \tau\right) = 0 \text{ if } x_n^m \neq x_{n'}^{\prime m'} \text{ and } \gamma\left(x_n^m, x_n^m, \tau\right) = \sigma\left(x_n^m\right)^2 \delta(\tau). \tag{14}$$

Interestingly, their computation led to both a mean field and a covariance matrix significantly different from those found in the previous PSV data analysis. In particular, significant differences were found between some $\sigma\ (g_n^m)$ and $\sigma\ (h_n^m)$ sharing the same degree n and order m, a property that is not compatible with a matrix of the form (13).

5 SYMMETRY CONDITIONS

As often noted in the early papers, but never discussed or proven in detail, it turns out that most of the previous assumptions and results regarding the forms $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}(\tau)$ may or should take can be interpreted in terms of symmetry conditions. Gubbins & Zhang (1993) already discussed the symmetry properties of the convective dynamo equations in a deterministic context. Here, however, we address the issue (and give complete proofs) in the more specific statistical context of GGP models and focus on the magnetic field one can observe.

First, consider a generalized GGP predicting the same statistical properties for the field at any location at the surface of the Earth, i.e. spherically symmetric. It can be shown (Appendix B) that the necessary and sufficient conditions for this to be the case is that, in any one frame $\Re$, $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}(\tau)$ satisfy:

Conditions for spherical statistical symmetry

$$\mu\left(g_n^m\right) = \mu\left(h_n^m\right) = 0, \tag{15a}$$

$$\gamma\left(x_n^m, x_{n'}^{\prime m'}, \tau\right) = 0 \text{ if } x_n^m \neq x_{n'}^{\prime m'} \quad \text{and} \quad \gamma\left(x_n^m, x_n^m, \tau\right) = \gamma_n(\tau). \tag{15b}$$

A field compatible with a GGP process satisfying eq. (15) would then be a field insensitive to any specific frame of reference.

Next consider a generalized GGP predicting exactly the same statistical properties at any two locations sharing the same colatitude at the surface of the Earth, i.e. axially symmetric about the geographical axis. It can be shown (Appendix A) that the necessary and sufficient conditions for this to be the case are now that, in any one frame $\Re$ with the z-axis being south–north, $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}(\tau)$ satisfy:

Conditions for axial statistical symmetry

$$\mu\left(g_n^m\right) = \mu\left(h_n^m\right) = 0 \qquad \text{if} \quad m \neq 0, \tag{16a}$$

$$\gamma\left(g_n^m, g_{n'}^m, \tau\right) = \gamma\left(h_n^m, h_{n'}^m, \tau\right) \qquad \text{if} \quad m \neq 0, \tag{16b}$$

$$\gamma\left(g_n^m, h_{n'}^m, \tau\right) = -\gamma\left(h_n^m, g_{n'}^m, \tau\right), \tag{16c}$$

$$\gamma\left(x_n^m, x_{n'}^{\prime m'}, \tau\right) = 0 \qquad \text{if} \quad m \neq m'. \tag{16d}$$

A field compatible with a GGP process satisfying eq. (16) would then be a field insensitive to any specific longitude.

Finally, consider the symmetry about the equatorial plane. The situation is slightly subtler, because individual SH functions are always either equatorial symmetric (E^{S}, using the terminology of Gubbins & Zhang 1993), or equatorial antisymmetric (E^{A}). That is, all $V = Y_n^{m(c,s)}(\theta, \varphi)$ with $(n - m)$ even satisfy (with $\mathbf{B} = -\nabla V$)

E^{S} *symmetry*

$$V(r, \theta, \varphi) = V(r, \pi - \theta, \varphi) \quad \text{and}$$
$$[B_r, B_\theta, B_\varphi](r, \theta, \varphi) = [B_r, -B_\theta, B_\varphi](r, \pi - \theta, \varphi); \tag{17}$$

whereas, all $V = Y_n^{m(c,s)}(\theta, \varphi)$ with $(n - m)$ odd satisfy

E^{A} *symmetry*

$$V(r, \theta, \varphi) = -V(r, \pi - \theta, \varphi) \quad \text{and}$$
$$[B_r, B_\theta, B_\varphi](r, \theta, \varphi) = [-B_r, B_\theta, -B_\varphi](r, \pi - \theta, \varphi). \tag{18}$$

Thus two types of equatorial symmetry ought to be considered. A GGP will be said to be E^{S} symmetric if it remains invariant after reflecting in the equatorial plane and changing the polarity, and E^{A} symmetric if it does so after just reflecting in the equatorial plane. [Note that the magnetic field $\mathbf{B} = -\nabla V$ being a pseudo-vector, its potential V is a pseudo-scalar. Upon physical reflection in the equatorial plane, a sign change therefore occurs: $V(r, \theta, \varphi)$ becomes $-V(r, \pi - \theta, \varphi)$ and not $V(r, \pi - \theta, \varphi)$, as would be the case if V had been a proper scalar, (see e.g. Gubbins & Zhang 1993).] It can then easily be checked that a GGP will be said to be E^{S} or E^{A} symmetric if and only if, in any one frame $\Re$ with the z-axis being south–north, $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}(\tau)$ satisfy:

Conditions for E^{S} *equatorial statistical symmetry*

$$\mu\left(g_n^m\right) = \mu\left(h_n^m\right) = 0 \qquad \text{if} \quad n - m \text{ is odd}, \tag{19a}$$

$$\gamma\left(x_n^m, x_{n'}^{\prime m'}, \tau\right) = 0 \qquad \text{if} \quad n - m \text{ and } n' - m' \text{ are of different parities}, \tag{19b}$$

or

Conditions for E^{A} *equatorial statistical symmetry*

$$\mu\left(g_n^m\right) = \mu\left(h_n^m\right) = 0 \qquad \text{if} \quad n - m \text{ is even}, \tag{20a}$$

$$\gamma\left(x_n^m, x_{n'}^{\prime m'}, \tau\right) = 0 \qquad \text{if} \quad n - m \text{ and } n' - m' \text{ are of different parities}. \tag{20b}$$

Finally, we will state that a GGP has equatorial symmetry if it has either E^{S} or E^{A} symmetries.

Then, all fields compatible with a GGP with an equatorial symmetry (i.e. satisfying 19 or 20) and only those, would be fields incapable of statistically distinguishing the Northern from the Southern Hemisphere. This can easily be checked, bearing in mind the important additional property that the global polarity of the field cannot itself be taken as a preference for either hemisphere. This is because of the well-known property of rotating fluid dynamos, which states that if such a dynamo can produce a field $\mathbf{B}(\mathbf{r}, t)$, then the very same dynamo (defined by exactly the same boundary conditions and non-magnetic time-varying quantities, such as temperature field, velocity field, etc.) can also produce the exact opposite field $-\mathbf{B}(\mathbf{r}, t)$ (see e.g. Gubbins & Zhang 1993).

In terms of the alternative terminology we recalled earlier (and more often used in palaeomagnetism, see e.g. Merrill *et al.* 1996) of dipole family ($n - m$ odd) versus quadrupole family ($n - m$ even), we thus conclude that a GGP will have equatorial symmetry if and only if (i) its mean field exclusively belongs to either the dipole or the quadrupole family, and (ii) the dipole family and quadrupole family components of the fluctuating field behave independently.

6 SYMMETRIES IN GGP MODELS

Reconsider now the assumptions and results reviewed in Section 4, in terms of symmetry properties. The case of the mean field $\boldsymbol{\mu}$ is trivial. All models involve non-zero mean fields and break the spherical symmetry (contradiction with eq. 15a). Early models only involve zonal mean fields and satisfy the axial symmetry about the rotation axis of the Earth (compatibility with eq. 16a). However, those zonal fields all involve at least a quadrupole axial contribution (with E^{S} symmetry) in addition to the axial dipole (with E^{A} symmetry), which prevents the whole mean field from having any of the two symmetries. They therefore break the equatorial symmetry. Finally, some of the most recent mean models seem to require non-zonal terms. This would involve axial symmetry breaking (contradiction with eq. 16a).

Consider now the less trivial case of the covariance matrix $\boldsymbol{\gamma}(\tau)$. The first set of assumptions introduced, eq. (10), exactly corresponds to eq. (15b). Thus, the early GGP models simply assumed that the way the MF fluctuates would not be sensitive to any specific frame of reference and would comply with spherical symmetry. The fact that the PSV data conflict with eq. (10) whatever the functions $\gamma_n(\tau)$, as shown by Hulot & Gallet (1996), can then be interpreted as the proof that this is not the case, that the fluctuations of the field also break the spherical symmetry and that GGP models must account for that fact.

Consider then eq. (13) introduced by Kono & Tanaka (1995). These assumptions leave the possibility for the covariance matrix to break the spherical symmetry, because one may have $\gamma(x_n^m, x_n^m, \tau) \neq \gamma(x'^{m'}_n, x'^{m'}_n, \tau)$ if $m \neq m'$, which conflicts with eq. (15b). On the other hand, it is easy to check that eq. (13) is compatible with both the axial and the equatorial symmetry conditions (16b–d) and (19b) or (20b). However, the reverse statement is not true and conditions (13) are therefore more restrictive than needed if it indeed turns out that the fluctuating field can break the spherical, but not necessarily the axial and/or equatorial symmetries.

Next consider the proposal of Kono *et al.* (2000) that eq. (14) rather than eq. (13) be used. If, as they tentatively suggest, some differences are to be found between the $\sigma(g_n^m)$ and $\sigma(h_n^m)$ sharing the same degree n and order m, then eq. (16b) would be violated and the axial symmetry no longer satisfied. However, eq. (14) only relaxes the constraint (16b), whereas axial symmetry breaking would also imply simultaneously relaxing eqs (16c) and (16d). This would then require that non-zero cross-covariances also be considered in $\boldsymbol{\gamma}(\tau)$, as originally proposed by Hulot & Gallet (1996).

Finally, it should be noted that although eq. (14) breaks the axial symmetry it assumes an equatorial symmetry (compatibility with eqs 19b or 20b). This we note is somewhat at odds with the fact that the mean field is found to break the equatorial symmetry (as testified by the need to introduce a zonal quadrupole in μ). To possibly account for a similar equatorial symmetry breaking in the fluctuating part of the field, non-zero cross-covariance would again need to be considered in $\boldsymbol{\gamma}(\tau)$ (at least between coefficients not belonging to the same family).

From all those studies, it thus appears that clear indications are found for spherical, equatorial and perhaps axial symmetry breaking in both $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}(\tau)$. However, inadequate assumptions with respect to the form $\boldsymbol{\gamma}(\tau)$ may take have often been made. It would be advisable to only rely on assumptions with clear symmetry meaning, such as eqs (15), (16), (19) and (20).

7 GEOPHYSICAL IMPLICATION

Relying on assumptions with clear symmetry meaning would make it possible to better identify the symmetries the field is indeed willing to brake, which would in turn provide us with some useful information about the way the geodynamo works and is possibly influenced by non-homogeneous boundary conditions. It is well known that the response of a physical system has symmetry that is either the same as or lower than that of the system itself (e.g. Gubbins & Zhang 1993). In the present case, the system is the rotating core with boundary conditions imposed on the corotating CMB. Because of the rotation, this system does not have a spherical symmetry and spherical symmetry breaking has to occur. The fact that in all recent GGP models, not only $\boldsymbol{\mu}$ but also $\boldsymbol{\gamma}(\tau)$ breaks this symmetry (and thus senses the rotation axis of the Earth) shows that this is indeed the case and that symmetry breaking can be detected in GGP models.

Axial and equatorial symmetry breaking results can bring additional information but require a more careful interpretation. Two possibilities are to be considered. One is that the CMB conditions are in fact symmetric and that symmetry breaking occurs only because the dynamo spontaneously takes advantage of the possibility it has to produce a field with lower symmetry than that of the system. The other is that CMB conditions are not homogeneous, significantly break the symmetries and therefore force the dynamo to produce a field breaking the symmetries in the same way. Which interpretation one should go for is not so obvious. However, useful suggestions can be made.

First, we note that spontaneous axial symmetry breaking under axisymmetrical CMB conditions is unlikely to occur. In such conditions, it indeed seems difficult for the system to keep a field (and all its characteristics) statistically fixed in longitude, given the possibility there is to rotate the whole system through infinitesimal steps about the rotation axis of the Earth. Results derived from numerical simulation clearly support this point of view (see Paper II). If confirmed (but see McElhinny *et al.* 1996; Carlut & Courtillot 1998), axial symmetry breaking, such as the one tentatively found in the mean field by several authors (Gubbins & Kelly 1993; Johnson & Constable 1997), would thus almost certainly testify for axial symmetry breaking in the CMB conditions.

By contrast to the previous case, we note that even if CMB conditions are symmetric about the equator, no such continuous way of exchanging the Northern and Southern Hemispheres can exist, making it less obvious for a dynamo solution displaying a statistical preference for one hemisphere to shift to the analogous state showing a preference for the opposite hemisphere. This situation would be analogous to the one encountered with the global polarity of the field, which can remain stable over long periods of time (or even forever, for some parameter regimes) even under homogeneous CMB conditions, despite the fact already pointed out that a dynamo has no reason to prefer one polarity to another. Thus, a dynamo could easily remain locked in an equatorial symmetry breaking state, even if the CMB conditions are symmetric about the equator. In this respect, equatorial symmetry breaking by the field, such as the one implied by the need to introduce an axial quadrupole mean field $\mu(g_2^0)$ in all recent GGP models, cannot as such be taken as evidence of equatorial symmetry breaking by the CMB conditions. Again, results derived from numerical simulation would support this point of view (see Paper II).

However, more can be said. Consider an equatorial symmetry breaking state, state I, as sketched on Fig. 1(a). In such a state, it is important to point out that not only the magnetic field, but also all convection-related fields (temperature, flows, etc.) would have to break the equatorial symmetry (Gubbins & Zhang 1993). If that state was to arise as a result of a spontaneous locking of the geodynamo under equatorial symmetric CMB conditions, any of the three following states would be fully equivalent to it: state II, which would correspond to state I after a polarity transition changing $\mathbf{B}(\mathbf{r}, t)$ into $-\mathbf{B}(\mathbf{r}, t)$ (and leaving the convection pattern unchanged; Fig. 1b); state III, corresponding to state I after reflection in the equatorial plane (i.e. a mirror transition, which would then leave CMB conditions invariant, but would change both the magnetic and convection patterns; Fig. 1c); and state IV, corresponding to state I after both transitions (Fig. 1d). By contrast, if state I was to arise because of equatorial symmetry breaking CMB conditions, no mirror transition would be possible and only state II would be fully equivalent to state I.

Characterizing state I by $[\boldsymbol{\mu}_{\mathrm{I}}, \gamma_{\mathrm{I}}(\tau)]$, this means that under equatorial symmetric CMB conditions, any of the three following transitions {to either state II, III, or IV characterized by $[\boldsymbol{\mu}_{\mathrm{II}}, \gamma_{\mathrm{II}}(\tau)]$, $[\boldsymbol{\mu}_{\mathrm{III}}, \gamma_{\mathrm{III}}(\tau)]$ and $[\boldsymbol{\mu}_{IV}, \gamma_{IV}(\tau)]$}, could potentially be observed:

(i) I→II transition, $\boldsymbol{\mu}_{\mathrm{II}} = -\boldsymbol{\mu}_{\mathrm{I}}$ and $\gamma_{\mathrm{II}}(\tau) = \gamma_{\mathrm{I}}(\tau)$ involving a full field reversal, but no change in the convective pattern;

(ii) I→III transition,

$$\mu_{\mathrm{III}}\left(x_n^m\right) = -(-1)^{(n-m)}\mu_{\mathrm{I}}\left(x_n^m\right) \quad \text{and}$$
$$\gamma_{\mathrm{III}}\left(x_n^m, x_{n'}^{\prime m'}, \tau\right) = (-1)^{(n-m)+(n'-m')}\gamma_{\mathrm{I}}\left(x_n^m, x_{n'}^{\prime m'}, \tau\right)$$

involving a quadrupole family field reversal (but leaving the dipole family field unchanged; recall that the magnetic field and potential are pseudo-vector and pseudo-scalar) and a mirror reversal of the convective pattern; or

(iii) I→IV transition,

$$\mu_{IV}\left(x_n^m\right) = (-1)^{(n-m)}\mu_{\mathrm{I}}\left(x_n^m\right) \quad \text{and}$$
$$\gamma_{IV}\left(x_n^m, x_{n'}^{\prime m'}, \tau\right) = (-1)^{(n-m)+(n'-m')}\gamma_{\mathrm{I}}\left(x_n^m, x_{n'}^{\prime m'}, \tau\right)$$

involving a dipole family field reversal (but leaving the quadrupole family field unchanged) and a mirror reversal of the convective pattern;

and that observing a lack of I→III or I→IV transitions, while many I→II transitions occur, would testify for the fact that the dynamo shows a very strong stability with respect to mirror reversals of the convective pattern (the one feature in common in I→III and I→IV transitions). This could then only occur either because of a strong spontaneous stability of the convective pattern (stronger than that of the magnetic polarity), or because the CMB conditions indeed significantly break the equatorial symmetry.

Both I→II and I→IV transitions would correspond to a sign change of $\mu(g_1^0)$, i.e. to what is usually taken as the definition of a field reversal. However, only I→II transitions would also involve a simultaneous sign change of $\mu(g_n^0)$ with n even (quadrupole family component) and of $\mu(g_2^0)$ in particular. I→III transitions would be harder to look for. Those transitions would not involve a sign change of the mean axial dipole $\mu(g_1^0)$ and would thus be difficult to identify in the palaeomagnetic record. However, those I→III transitions could occur during so-called excursions, when the dynamo moves away from an initial state with a given $\mu(g_1^0)$ and finally settles back to another state with the same $\mu(g_1^0)$. Searching for a possible sign change in $\mu(g_2^0)$ not only at times of reversals but also at times of excursions could therefore also prove interesting.

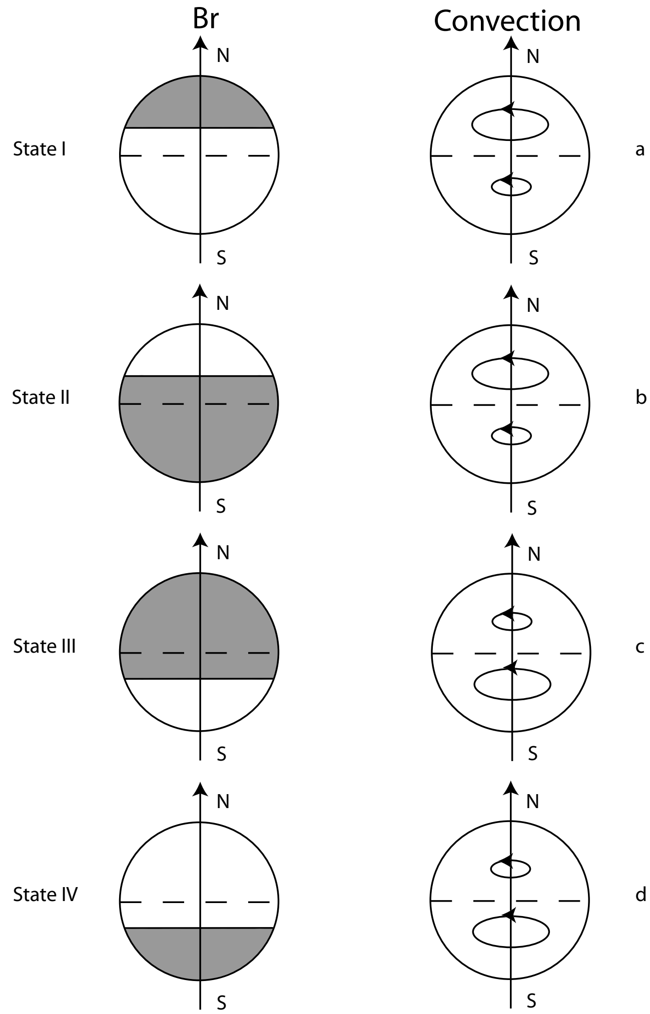

Figure 1. An equatorial symmetry breaking state I displays dissymmetry not only in its magnetic state but also in its convection state (a). If core–mantle boundary (CMB) conditions are symmetric with respect to the equator, from the point of view of the system, this state is equivalent to any of the three following states: (b) state II, obtained from state I by simply going through a polarity transition changing $\mathbf{B}(\mathbf{r}, t)$ into $-\mathbf{B}(\mathbf{r}, t)$ (but leaving the convection pattern unchanged); (c) state III, obtained from state I by reflection in the equatorial plane, (a mirror transition, changing both the magnetic and convection patterns); and (d) state IV, corresponding to state I after both transitions. Going from state I to state II, would involve a full field reversal, but no change in the convective pattern; going from state I to state III would involve a quadrupole family field reversal and a mirror reversal of the convective pattern; going from state I to state IV would finally involve a dipole field reversal and a mirror reversal of the convective pattern. If the CMB conditions are not symmetric with respect to the equator, no mirror transition is possible and the system can only go from state I to state II.

Separate analysis of the data for normal and reverse polarity shows that a simultaneous sign change of $\mu(g_1^0)$ and $\mu(g_2^0)$ occurred during the last reversal (Bruhnes–Matuyama). This reversal must have been a I→II transition. Combined analysis of all recent (0–5 Myr) normal data on one hand and of all reverse data on another hand also lead to average fields that would argue in favour of reversals being systematically of that type (McElhinny *et al.* 1996; Johnson & Constable 1997; Carlut & Courtillot 1998; Hatakeyama & Kono 2002). If that is indeed the case, we note that this would then also mean that some coupling between the dipole and quadrupole family components of the field necessarily took place, in particular at times of reversal, as first envisioned by Merrill & McFadden (1988). However, such combined analysis could be biased by the fact that more than half of the data actually belongs to those two last chrons (Quidelleur *et al.* 1994; Johnson & Constable 1996). As a matter of fact, Schneider & Kent (1988) noted early on that some variations could occur in the value the ratio $\mu(g_2^0)/\mu(g_1^0)$ takes, depending on the chron considered. Schneider & Kent (1988) interpreted this as a sign that some standing non-dipole field could be present at times of reversals. However, it is tempting to argue that this could also be interpreted as an indication that some quadrupole family field reversals could have taken place during those chrons. This interpretation would also be consistent with the claim made a little later by the same authors (Schneider & Kent 1990) that, by contrast, the zonal octupole mean field component they found in their data analysis has a constant $\mu(g_3^0)/\mu(g_1^0)$ ratio. However, because all those results have since been dismissed (see e.g. McElhinny *et al.* 1996), they would clearly need to be double-checked by, for example, building models for individual chrons (and before and after excursions), and testing whether I→III and I→IV transitions could have possibly occurred. Until being possibly proven otherwise, it however seems reasonable to argue that the current data rather suggest a lack of any other transition than I→II, hence a lack of mirror transitions in favour of a strong stability of the convection pattern.

This finally prompts the issue of such stability possibly being a spontaneous feature of a dynamo under equatorial symmetric CMB conditions. This issue could soon be settled with the help of numerical simulation of dynamos run under such CMB conditions. When long runs displaying many successive reversals will be made available, it will become possible to search for mirror transitions (and I→III and I→IV transitions). If it turns out that such transitions can occur in those simulations, then only, the suggested palaeomagnetic evidence for a $\mu(g_2^0)$ reversing sign with $\mu(g_1^0)$ at times of reversals could be taken as serious evidence of equatorial symmetry breaking in the CMB.

8 CONCLUSION

Several conclusions can be drawn from the present study. First, that in attempts to find GGP models best describing the field over the past 5 Myr, most of the simplifying assumptions chosen so far turn out to correspond to strong symmetry assumptions. Secondly, that the failure of some of the early models to account for the data and the consequent need to introduce some complexity in those models can readily be explained in terms of symmetry breaking properties of the field. Thirdly, that such symmetry breaking properties can bring important information about the way the dynamo works and is possibly influenced by the CMB conditions.

We also noted that equatorial symmetry breaking states can exist even if CMB conditions display equatorial symmetry and that such states can potentially go through both polarity and mirror transitions, possibly leading to either a full field, a dipole family field, or a quadrupole family field reversal. Thus, to prove that an equator symmetry breaking state (such as the one that seems to have characterized the recent palaeomagnetic field) is evidence for symmetry breaking CMB conditions, two conditions need to be satisfied: (i) that no mirror transition (hence neither dipole family field nor quadrupole family field reversals) be observed in the data and (ii) that by contrast such mirror transitions be observed in numerical simulations with equatorial symmetric CMB conditions. A careful comparison of the statistical behaviour of the field before and after both reversals and excursions could prove very useful to check (i). Future long runs from dynamo simulations could help address (ii).

Only evidence of axial symmetry breaking field properties could otherwise establish the reality of some influence of inhomogeneous CMB conditions on the geodynamo. Evidence for such symmetry breaking properties has also been tentatively found in the recent palaeomagnetic field, but only in the mean field $\boldsymbol{\mu}$ so far. The present study further suggests that additional evidence could possibly be found in studying the covariance matrix $\boldsymbol{\gamma}(\tau)$. This would however require that *a priori* simplifying assumptions of the type (10), (13) or (14), be abandoned and that the form $\boldsymbol{\gamma}(\tau)$ is spontaneously willing to take in such instances be investigated. This is possible thanks to numerical simulation. In fact, those simulations can more generally be used to (i) assess the global validity of the generalized GGP approach, (ii) define the best simplifications one may use in $\boldsymbol{\gamma}(\tau)$, (iii) investigate the symmetry breaking issues we raised in the present paper, all this under well-controlled conditions. This is being done in Paper II, devoted to the analysis of numerical simulations from the Glatzmaier & Roberts (1995, 1996, 1997; Glatzmaier *et al.* 1999) dynamo.

ACKNOWLEDGMENTS

We thank Cathy Constable and an anonymous reviewer for their constructive comments, which substantially helped improve the original manuscript. This research was completed while GH held a National Research Council Research Associateship Award at NASA Goddard Space Flight Center. Institut de Physique du Globe de Paris Contribution no. 2039 and Institut National des Sciences de l'Univers - Centre National de la Recherche Scientifique Contribution no. 377.

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APPENDIX A: DERIVATION OF THE CONDITIONS FOR AXIAL STATISTICAL SYMMETRY

Let us first recall the definition of the Schmidt quasi-normalized SH functions used to define $V(\mathbf{r}, t)$ with the help of eq. (2) (see e.g. Langel 1987):

$$Y_n^{mc}(\theta, \varphi) = P_n^m(\cos\theta)\cos m\varphi; \quad Y_n^{ms}(\theta, \varphi) = P_n^m(\cos\theta)\sin m\varphi; \tag{A1}$$

where $n \geq 1$ and $0 \leq m \leq n$, and the associate Legendre functions $P_n^m(u)$ are then defined by

$$P_n^m(u) = \left[\frac{2(n-m)!}{(n+m)!}\right]^{1/2}(1-u^2)^{m/2}\frac{\mathrm{d}^m P_n(u)}{\mathrm{d}u^m} \qquad \text{for} \quad m > 0, \tag{A2}$$

$$P_n^0(u) = P_n(u) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}u^n}(u^2-1)^n, \tag{A3}$$

where $P_n(u)$ is the Legendre polynomial of degree n.

Let us next introduce the following useful alternative complex representation of the magnetic potential $V(\mathbf{r}, t)$:

$$V(\mathbf{r}, t) = a\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{n+1}\sum_{m=-n}^{n} b_n^m(t)Y_n^m(\theta, \varphi), \tag{A4}$$

where the complex SH function $Y_n^m(\theta, \varphi)$ is now defined by

$$Y_n^m(\theta,\varphi) = P_n^m(\cos\theta)e^{im\varphi} \qquad \text{for} \quad -n \le m \le n \tag{A5}$$

and the definition of P_n^m (cos θ) is extended to negative values of m with the help of

$$P_n^{-m}(u) = (-1)^m P_n^m(u) \qquad \text{for} \quad -n \le m \le n. \tag{A6}$$

Because the potential $V(\mathbf{r}, t)$ is a real function, it can easily be checked that the complex Gauss coefficients $b_n^m(t)$ must satisfy

$$b_n^{m^*} = (-1)^m b_n^{-m}, \tag{A7}$$

where the star refers to the complex conjugate.

Identifying eqs (A4) and (2) then leads to the identities

$$b_n^m = \left[g_n^m - i h_n^m\right]\left(\frac{1+\delta_{m,0}}{2}\right) \qquad \text{for} \quad 0 \le m \le n \tag{A8}$$

(using eq. A7 makes it possible to derive a similar identity in the case $m < 0$) and

$$g_n^0 = b_n^0 \qquad \text{and} \qquad \begin{cases} g_n^m = b_n^m + (-1)^m b_n^{-m} \\ h_n^m = i\left[b_n^m - (-1)^m b_n^{-m}\right] \end{cases} \qquad \text{for} \quad 0 < m \le n. \tag{A9}$$

If we then consider a rotation of φ_0 of the frame of reference about the initial Oz axis, it is quite straightforward to check that, in the new frame of reference, the real [$g_n^m(\varphi_0)$, $h_n^m(\varphi_0)$] and complex [$b_n^m(\varphi_0)$] Gauss coefficients are related to the Gauss coefficients in the original frame of reference (g_n^m, h_n^m, b_n^m) through:

$$b_n^m(\varphi_0) = b_n^m e^{im\varphi_0}, \tag{A10}$$

$$\begin{cases} g_n^m(\varphi_0) = g_n^m \cos m\varphi_0 + h_n^m \sin m\varphi_0 \\ h_n^m(\varphi_0) = -g_n^m \sin m\varphi_0 + h_n^m \cos m\varphi_0. \end{cases} \tag{A11}$$

Now let us assume the magnetic field can be defined as being the result of a generalized GGP, so that the real Gauss coefficients [$g_n^m(t)$, $h_n^m(t)$] define a vector $\mathbf{x}(t)$, which can be described in terms of a realization of a multidimensional stationary random Gaussian process. This process is then defined by eqs (3) and (4). Alternatively, the same process can also be defined by the more convenient (but equivalent) complex quantities:

$$E\left\{b_n^m(t)\right\} = \beta_n^m \qquad \text{and} \qquad E\left\{\left[b_n^m(t) - \beta_n^m\right]\left[b_{n'}^{m'}(t') - \beta_{n'}^{m'}\right]^*\right\}, \tag{A12}$$

where we have introduced the notation β_n^m rather than $\mu(b_n^m)$ to ease notations in what follows.

After a change of frame of reference by rotation of φ_0 about Oz, those quantities become

$$E\left\{b_n^m(\varphi_0, t)\right\} = \beta_n^m(\varphi_0) \quad \text{and} \quad E\left\{\left[b_n^m(\varphi_0, t) - \beta_n^m(\varphi_0)\right]\left[b_{n'}^{m'}(\varphi_0, t') - \beta_{n'}^{m'}(\varphi_0)\right]^*\right\}. \tag{A13}$$

For the process to be axially symmetric about the geographical axis and predict exactly the same statistical properties at any two locations sharing the same colatitude at the surface of the Earth, the necessary and sufficient conditions to be satisfied are then that:

$$\beta_n^m(\varphi_0) = \beta_n^m, \tag{A14a}$$

$$E\left\{\left[b_n^m(\varphi_0, t) - \beta_n^m(\varphi_0)\right]\left[b_{n'}^{m'}(\varphi_0, t') - \beta_{n'}^{m'}(\varphi_0)\right]^*\right\} = E\left\{\left[b_n^m(t) - \beta_n^m\right]\left[b_{n'}^{m'}(t') - \beta_{n'}^{m'}\right]^*\right\}, \tag{A14b}$$

whatever the value of φ_0.

From eq. (A10) it is quite obvious that eq. (A14a) implies that

$$\beta_n^m = 0 \qquad \text{if} \quad m \neq 0, \tag{A15a}$$

which is equivalent to eq. (16a) because of eqs (A7), (A8) and (A9).

Again, because of eq. (A10), it is straightforward to check that eq. (A14b) with eqs (A14a) and (A15a) then implies

$$E\left\{\left[b_n^m(t) - \beta_n^m\right]\left[b_{n'}^{m'}(t') - \beta_{n'}^{m'}\right]^*\right\} = 0 \qquad \text{if}\, m \neq m'. \tag{A15b}$$

In the case $m \ge 0$, $m' \ge 0$, making use of eq. (A8) in eq. (A15b) then leads to:

if $m \neq m'$,

$$\begin{cases} E\left\{\left[g_n^m(t) - \mu(g_n^m)\right]\left[g_{n'}^{m'}(t') - \mu(g_{n'}^{m'})\right]\right\} = -E\left\{\left[h_n^m(t) - \mu(h_n^m)\right]\left[h_{n'}^{m'}(t') - \mu(h_{n'}^{m'})\right]\right\}, & \text{(A16a)} \\ E\left\{\left[g_n^m(t) - \mu(g_n^m)\right]\left[h_{n'}^{m'}(t') - \mu(h_{n'}^{m'})\right]\right\} = E\left\{\left[h_n^m(t) - \mu(h_{n'}^{m'})\right]\left[g_{n'}^{m'}(t') - \mu(g_{n'}^{m'})\right]\right\}. & \text{(A16b)} \end{cases}$$

Next, setting $m' \to -m'$ in eq. (A15b) and using eq. (A7) leads to

$$E\left\{\left[b_n^m(t) - \beta_n^m\right]\left[b_{n'}^{m'}(t') - \beta_{n'}^{m'}\right]\right\} = 0 \qquad \text{if} \quad m \neq -m'. \tag{A15c}$$

Again in the case $m \geq 0$, $m' \geq 0$, making use of eq. (A8) in eq. (A15c) finally leads to:

if $(m, m') \neq (0, 0)$,

$$\begin{cases} E\left\{\left[g_n^m(t) - \mu\left(g_n^m\right)\right]\left[g_{n'}^{m'}(t') - \mu\left(g_{n'}^{m'}\right)\right]\right\} = E\left\{\left[h_n^m(t) - \mu\left(h_n^m\right)\right]\left[h_{n'}^{m'}(t') - \mu\left(h_{n'}^{m'}\right)\right]\right\} & \text{(A17a)} \\ E\left\{\left[g_n^m(t) - \mu\left(g_n^m\right)\right]\left[h_{n'}^{m'}(t') - \mu\left(h_{n'}^{m'}\right)\right]\right\} = -E\left\{\left[h_n^m(t) - \mu\left(h_n^{m'}\right)\right]\left[g_{n'}^{m'}(t') - \mu\left(g_{n'}^{m'}\right)\right]\right\}. & \text{(A17b)} \end{cases}$$

Combining eq. (A16) with eq. (A17) and recalling eq. (4) then leads to eqs (16b), (16c), (16d) with $\tau = (t' - t)$.

Conversely, it is quite straightforward to check that eqs (16b), (16c), (16d) imply eq. (A15b) and thus eq. (A14b) (the case $m = m'$ being trivial). Thus, eq. (16) is the necessary and sufficient conditions to be satisfied in anyone frame $\Re$ with z-axis being south–north for the process to predict exactly the same statistical properties at any two locations sharing the same colatitude at the surface of the Earth.

APPENDIX B: DERIVATION OF THE CONDITIONS FOR SPHERICAL STATISTICAL SYMMETRY

Let us first consider the way Gauss coefficients are transformed after a rotation of θ_0 of the frame of reference about its Oy axis. The angular coordinates (θ, φ) [respectively (θ', φ')] of a point in the initial (respectively final) frame of reference, satisfy:

$$\begin{cases} \sin\theta\sin\varphi = \sin\theta'\sin\varphi' \\ \cos\theta = \cos\theta'\cos\theta_0 - \sin\theta'\cos\varphi'\sin\theta_0 \\ \sin\theta\cos\varphi = \cos\theta'\sin\theta_0 + \sin\theta'\cos\varphi'\cos\theta_0 \end{cases}. \tag{B1}$$

In such a case, the following theorem of addition for the complex SH functions applies:

$$a_m Y_n^m(\theta, \varphi) = \sum_{k=-n}^{n} a_k P_{mk}^n(\cos\theta_0) Y_n^k(\theta', \varphi'), \tag{B2}$$

where the $Y_n^m(\theta, \varphi)$ are defined by eqs (A5), (A2), a_m is defined by

$$a_m = (i)^m \sqrt{\frac{1 + \delta_{m,0}}{2}} \tag{B3}$$

and the $P_{mk}^n(u)$ functions are defined by

$$P_{mk}^n(u) = \frac{(-1)^{n-k}(i)^{k-m}}{2^n}\sqrt{\frac{(n+m)!}{(n-k)!(n+k)!(n-m)!}} \times (1+u)^{-\left(\frac{m+k}{2}\right)}(1-u)^{\frac{k-m}{2}}\frac{\mathrm{d}^{n-m}}{\mathrm{d}u^{n-m}}\left[(1-u)^{n-k}(1+u)^{n+k}\right]. \tag{B4}$$

[We derived eq. (B2) with the help of Vilenkin (1969). It follows from eq. (3) of III-4-2 in Vilenkin (1969), given that eq. (B1) amounts to eqs (6) and (6)′ of III-4-1 in Vilenkin (1969). Eq. (B4) is the same definition of $P_{mk}^n(u)$ as eq. (3) of III-3-4 in Vilenkin (1969).] The a_m factors defined by eq. (B3) arise because of our definition (A5) of the $Y_n^m(\theta, \varphi)$, which relies on the definition (A2) of the $P_n^m(u)$, normalized differently than the associate Legendre functions in Vilenkin (1969).

Relying on eq. (B2), it is then straightforward to check that the complex Gauss coefficients $b_n^m(\theta_0)$ in the new frame of reference are related to those b_n^m (in the original frame of reference) through

$$b_n^k(\theta_0) = \sum_{m=-n}^{n} b_n^m \frac{a_k}{a_m} P_{mk}^n(\cos\theta_0). \tag{B5}$$

As in Appendix A, let us now assume that the magnetic field can be defined as being the result of a generalized GGP, defined by a multidimensional stationary random Gaussian process $\mathbf{x}(t)$, satisfying eqs (3) and (4), equivalent to eq. (A12).

Let us further assume that the process is spherically symmetric and predicts the same statistical properties at any location at the surface of the Earth. In particular, this implies that the process is axially symmetric about the geographical axis. We may thus already conclude that eqs (A15)–(A17) and their consequences eq. (16) all apply again in the present case.

However, now we request more. In particular, we further request the statistics to remain invariant after any rotation of θ_0 about the Oy axis of the original frame of reference. Introducing

$$E\left\{b_n^m(\theta_0, t)\right\} = \beta_n^m(\theta_0), \tag{B6}$$

this then implies that, whatever θ_0:

$$\beta_n^m(\theta_0) = \beta_n^m, \tag{B7a}$$

$$E\left\{\left[b_n^m(\theta_0, t) - \beta_n^m(\theta_0)\right]\left[b_{n'}^{m'}(\theta_0, t') - \beta_{n'}^{m'}(\theta_0)\right]^*\right\} = E\left\{\left[b_n^m(t) - \beta_n^m\right]\left[b_{n'}^{m'}(t') - \beta_{n'}^{m'}\right]^*\right\}. \tag{B7b}$$

As far as the β_n^m are considered, because we already know that eq. (A15a) must be satisfied, we only need to deal with the special case $m = 0$. From eqs (B5) and (A15a), we infer

$$\beta_n^0(\theta_0) = \beta_n^0 P_{00}^n(\cos\theta_0) = \beta_n^0 P_n(\cos\theta_0). \tag{B8}$$

However, we know that P_n (cos θ_0) is not unity for all θ_0. Hence, eq. (B8) with eq. (B7a) imply $\beta_n^0 = 0$, so that, given eq. (A15a),

$$\beta_n^m = 0 \quad \forall(n, m), \tag{B9}$$

which is equivalent to eq. (15a) because of eqs (A7), (A8) and (A9).

Now consider the consequences of eq. (B7b). We already know that eq. (A15b) applies and it thus only remains to consider the case $m = m'$. Eqs (B5) and (B9) imply

$$E\left\{b_n^k(\theta_0, t)b_{n'}^{k*}(\theta_0, t')\right\} = \sum_{m=-n}^{n} \left|\frac{a_k}{a_m}\right|^2 P_{mk}^n(\cos\theta_0)P_{mk}^{n'*}(\cos\theta_0)E\left\{b_n^m(t)b_{n'}^{m^*}(t')\right\}. \tag{B10}$$

Integrating eq. (B10) over θ_0, taking advantage of the orthogonality property (see eq. 7 of III-6-2 in Vilenkin 1969),

$$\int_{-1}^{1} P_{mk}^n(u)P_{mk}^{n'*}(u)\,\mathrm{d}u = \frac{2}{2n+1}\delta_{nn'} \tag{B11}$$

and given eq. (B7b) with eq. (B9), we infer

$$E\left\{b_n^k(t)b_{n'}^{k*}(t')\right\} = \frac{1}{2}\sum_{m=-n}^{n}\left|\frac{a_k}{a_m}\right|^2 \frac{2}{2n+1}\delta_{nn'}E\left\{b_n^m(t)b_{n'}^{m^*}(t')\right\}. \tag{B12}$$

If $n \neq n'$, this leads to

$$E\left\{b_n^k(t)b_{n'}^{k*}(t')\right\} = 0 \qquad \text{if} \quad n \neq n'. \tag{B13}$$

If $n = n'$, eq. (B12) implies

$$\frac{1}{|a_k|^2}E\left\{b_n^k(t)b_n^{k*}(t')\right\} = \frac{1}{2n+1}\sum_{m=-n}^{n}\frac{1}{|a_m|^2}E\left\{b_n^m(t)b_n^{m^*}(t')\right\}. \tag{B14}$$

Because the right-hand side of eq. (B14) is independent of k, we may conclude that eq. (B14) (with eq. B3) implies

$$E\left\{b_n^m(t)b_n^{m^*}(t')\right\} = \frac{1+\delta_{m,0}}{2}F(n, t, t'), \tag{B15}$$

where $F(n, t, t')$ is a function of n, t, t'. (Note that eq. B15 then implies eq. B14.) It can easily be checked with the help of eqs (B14) and (A7) that $F(n, t, t') = F^*(n, t, t')$, showing that $F(n, t, t')$ is a real function. All in all, we may thus conclude that in addition to eq. (B9) (equivalent to eq. 15a), we also have (because of eqs A15, B13 and B15)

$$E\left\{b_n^m(t)b_{n'}^{m'^*}(t')\right\} = \delta_{n,n'}\delta_{m,m'}\frac{1+\delta_{m,0}}{2}F(n, t, t'). \tag{B16}$$

Relying on eqs (A8) and (A9), it is then quite straightforward to check that eq. (B16) is equivalent to eq. (15b). Thus, spherical symmetry implies eqs (B9) and (B16), equivalent to eq. (15).

Finally, it is important to check that eq. (15) (or eqs B9 and B16) are also sufficient conditions for spherical symmetry. This requires to check that once eqs (B9) and (B16) are satisfied in a given frame of reference (say the standard latitude/longitude frame), the process would be defined by exactly the same mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\gamma}(t' - t)$ in any other frame of reference, hence after any type of rotation of the axis.

To prove this, we first check that eqs (B9) and (B16) imply eq. (B7), i.e. invariance of the process under any rotation about the *Oy* axis. Given eq. (B5), the case for eq. (B7a) is trivial. Proving eq. (B7b) is less trivial. Knowing that eq. (B7a) is satisfied, we need to calculate (taking eqs B9 and B5 into account)

$$E\left\{b_n^k(\theta_0, t)b_{n'}^{k'*}(\theta_0, t')\right\} = \sum_{m=-n}^{n}\sum_{m'=-n'}^{n'} E\left\{b_n^m(t)b_{n'}^{m'*}(t')\right\}\frac{a_k}{a_m}\frac{a_{k'}^*}{a_{m'}^*}P_{mk}^n(\cos\theta_0)P_{m'k'}^{n'*}(\cos\theta_0). \tag{B17}$$

Given eq. (B16) and taking advantage of the two following properties,

$$P_{mk}^n(u) = P_{km}^n(u) \tag{B18}$$

and

$$\sum_{m=-n}^{n} P_{km}^n(u)P_{k'm}^n(u) = \delta_{kk'} \tag{B19}$$

(eqs B18 and B19 are respectively eq. 5 of III-3-6 and eq. 11′ of III-4-1 in Vilenkin 1969), together with eq. (B3), it can be checked that eq. (B17) implies

$$E\left\{b_n^k(\theta_0, t)b_{n'}^{k'*}(\theta_0, t')\right\} = E\left\{b_n^k(t)b_{n'}^{k'*}(t')\right\}. \tag{B20}$$

Given eq. (B9), eq. (B20) is eq. (B7b).

Thus, it appears that eqs (B9) and (B16) (or equivalently eq. 15) are sufficient conditions for the process to be invariant under any rotation about the *Oy* axis. Because they also imply eq. (16), they also are sufficient conditions for the process to be invariant under any rotation about the *Oz* axis. It then remains to show that they also are sufficient conditions for the process to be invariant under any other type of rotation (i.e. change of frame of reference).

To show this, we rely on the well-known property that any rotation can always be defined by three Euler angles defining three successive elementary rotations leading to the same final transformation. Let us denote $\Re_1$ and $\Re_4$ the initial and final frames of reference. The rotation of $\Re_1$ into $\Re_4$ can be decomposed into a first finite rotation of $\Re_1$ about the Oz_1 axis, leading to an intermediate frame $\Re_2$, a second finite rotation of $\Re_2$ about the Oy_2 axis, leading to another intermediate frame $\Re_3$, and a final finite rotation of $\Re_3$ about the Oz_3 axis leading to $\Re_4$.

If eqs (B9) and (B16) are satisfied in $\Re_1$, it follows from our earlier results that the process is invariant under any rotation about the Oz_1 axis. Thus, the process is defined by the same $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}(\tau)$ in both $\Re_1$ and $\Re_2$. However, because eqs (B9) and (B16) (equivalent to eq. 15) make a definition of $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}(\tau)$, it follows that eqs (B9) and (B16) are also satisfied in $\Re_2$. This then shows that the process is invariant under any rotation about the Oy_2 axis, and is again defined by the same $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}(\tau)$ in $\Re_3$, where eqs (B9) and (B16) again hold. Hence, the process is again invariant under the rotation about the Oz_3 axis leading to $\Re_4$ where the process is finally defined by the same $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}(\tau)$ as in $\Re_1$.

It therefore appears that eqs (B9) and (B16), or alternately eq. (15), are the necessary and sufficient conditions to be satisfied in any one frame of reference for the process to be spherically symmetric.