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SUMMARY

We again consider the problem of recovering the Earth's internal magnetic field **B** knowing its intensity $\|\mathbf{B}\|$ at the Earth's surface and the location of the dip equator. In the present paper we focus on estimating the difference between two solutions **B** obtained from imperfect data. We explicitly estimate this difference and show that it converges to zero when the errors on $\|\mathbf{B}\|$ and on the location of the dip equator E both tend to zero.

Key words: Backus Effect, geomagnetic field, potential theory.

1 INTRODUCTION

In a recent paper (Khokhlov *et al.* 1997) we discussed theoretically the problem of recovering the Earth's internal magnetic field **B** from knowledge of only its intensity $\|\mathbf{B}\|$ at the Earth's surface. We showed that the solution of the problem is unique if, in addition to the knowledge of $\|\mathbf{B}\|$, the set of points E where **B** is tangent to the Earth's surface—the *dip equator*—is known. In practice, the intensity is measured within a certain accuracy—more importantly, the external field, or part of it, must be considered as an error when trying to recover the internal field—and the location of the dip equator is only approximately known. It is then necessary to take into account these uncertainties in the solution of the problem. In the present paper we focus on the estimate of the difference between two solutions **B** obtained from these imperfect data. We show that this difference converges to zero when the errors on $\|\mathbf{B}\|$ and on the location of the dip equator E both tend to zero. The way this difference converges to zero provides a nice theoretical explanation of the practical success of geomagnetic modelling based on intensity data and on some knowledge of the dip equator (Ultré-Guérard *et al.* 1998). We again adopt a mathematical language in this paper, as in our two earlier papers (Khokhlov *et al.* 1997; Hulot *et al.* 1997), devoted to the problem of theoretical uniqueness. For mathematical simplicity, all quantities are defined in non-dimensional units.

2 THE PROBLEM AND THE MAIN RESULT

Let us start with the list of notations, assumptions and conditions we use. Σ and Ω stand respectively for the convex smooth closed surface in $\mathbb{R}^3$ (where the intensity data $\|\mathbf{B}\|$ are collected) and the (unbounded) part of $\mathbb{R}^3$ outside Σ; $\mathbf{n}=\mathbf{n}(\mathbf{x})$ denotes the outward unit vector normal to Σ at $\mathbf{x}\in\Sigma$. Without loss of generality, we assume that the origin of Cartesian coordinates is well inside Σ; that is, not in some open domain $\Omega'\supset(\Omega\bigcup\Sigma)$.

Let us now define the class of harmonic potentials f (=class of harmonic vector fields **B** such that $\mathbf{B}=\nabla f$) that we consider.

2.1 Constraints

We restrict ourselves to the class $\mathscr{E}(\varepsilon,\delta)$ of harmonic potentials f in the open domain Ω' satisfying the following conditions.

Condition 1 for $\mathbf{x}\in\Omega'\subset\mathbb{R}^3$, $\quad |f(\mathbf{x})| < \left(\dfrac{K}{|\mathbf{x}|}\right)^d,$

K and d being positive constants, $d\geq 1$. For the geomagnetic applications we are mainly interested in, Σ can be taken to be the surface of the Earth, all sources being assumed to be well within Σ (i.e. outside Ω'). In the absence of magnetic monopoles, condition 1 therefore applies with $d=2$.

Turning to the definition of the dip equator and of the uncertainty concerning its location, we consider the normal component $B_n=\mathbf{B}\cdot\mathbf{n}$ of the harmonic vector field $\mathbf{B}=\nabla f$ over Σ, and assume

Condition 2 Σ being the union of three subsets U_ε^+, U_ε^-, U_ε^0 defined by some given $\varepsilon>0$, $B_n=\nabla f\cdot\mathbf{n}$ is such that:

(1) $\mathbf{x}\in U_\varepsilon^+,\quad B_n>\varepsilon,$

(2) $\mathbf{x}\in U_\varepsilon^-,\quad B_n<-\varepsilon,$

(3) $\mathbf{x}\in U_\varepsilon^0,\quad |B_n|\leq\varepsilon.$

This condition both provides an approximate location of the dip equator and gives information on the polarity of **B** (upward or downward). Note that we do not necessarily require the dip-equator to be a single curve.

Finally, with respect to the accuracy of the intensity data, we define an uncertainty $\delta/2$ on the determination of $\|\mathbf{B}\|$ over Σ, which we assume to be much smaller than the maximal value $B_{\max}$ of $\|\mathbf{B}\|$ over Σ (in other words, the difference between two measurements of intensity is assumed to be always less than δ):

Condition 3 for any two functions f and g belonging to the class $\mathscr{E}(\varepsilon, \delta)$ and for any $\mathbf{x} \in \Sigma$ we have

(1) $|\nabla f(\mathbf{x})| \le B_{\max}$,

(2) $|\nabla g(\mathbf{x})| \le B_{\max}$,

(3) $||\nabla f(\mathbf{x})| - |\nabla g(\mathbf{x})|| \le \delta \ll B_{\max}$.

2.2 Convergence theorem

The main result we intend to show is the following theorem:

Theorem 1 *Consider two (not necessarily different) potentials f and g belonging to the class $\mathscr{E}(\varepsilon, \delta)$. If $\varepsilon, \delta \to 0$ then $|\nabla f - \nabla g| \to 0$ in Ω.*

Note

In fact an even stronger result holds: if $\varepsilon, \delta \to 0$ then $|\nabla f - \nabla g| \to 0$ also at $\mathbf{x} \in \Sigma$. However, the proof of this separate statement is more elaborate and will therefore be given separately in the Appendix.

3 AUXILIARY RESULTS

It is known that for the class of potentials described above (and in fact for a much wider class of harmonic potentials) some estimate bearing on the gradient implies some estimate bearing on the potential itself, and vice versa. The corresponding formal statement has many different versions; we refer here to Corollary 5.18 of Mitrea (1994):

Lemma 2 *For any harmonic function u in Ω which tends to zero at infinity*

$$\iint_\Sigma |u|^2\, dS \sim \iiint_\Omega |\nabla u|^2 \operatorname{dist}(x, \Sigma)\, dV .$$

Here ($\sim$) means equivalence of the two quantities F and G depending on some parameter η; it signifies that there always exist two positive constants D_1 and D_2 such that, for all η, we have $F(\eta) < D_1 G(\eta)$ and $G(\eta) < D_2 F(\eta)$.

Lemma 3 *Let a harmonic potential u satisfy Condition 1 and at $\mathbf{x}_0 \in \Sigma$ achieve its maximum value $u(\mathbf{x}_0) = a > 0$ over $\Omega \bigcup \Sigma$. Then $|\nabla u(\mathbf{x}_0)| > \hat{C} a^{1+1/d} K^{-1}$, where $\hat{C}$ is a positive constant that can be taken equal to $1/9$ for $d \ge 2$ and $1/16$ for $2 > d \ge 1$*

Proof. Let us first consider the special case with $K = 1$ and $a = 1$. Let the point $\hat{\mathbf{x}}$ be the centre of the sphere Σ_1 of radius R_1 tangent at $\mathbf{x}_0$ to Σ. Since Σ is convex, $(\Sigma_1 \backslash \mathbf{x}_0) \subset \Omega$. Consider also the smaller sphere Σ_2 of radius R_2 with the same centre $\hat{\mathbf{x}}$. Obviously, for any point $\mathbf{x} \in \Sigma_2$ we have $|\mathbf{x}| \ge R_1 - R_2$. Therefore, by Condition 1, for any point $\mathbf{x} \in \Sigma_2$ we have $u(\mathbf{x}) < 1/[(R_1 - R_2)^d]$. Denote by Ω_{12} the part of $\mathbb{R}^3$ bounded by Σ_1 and Σ_2 (see Fig. 1).

Consider the harmonic function in Ω_{12}

$$v(\mathbf{x}) = 1 - u(\mathbf{x}) - \left(\frac{1}{|\mathbf{x} - \hat{\mathbf{x}}|} - \frac{1}{R_1}\right).$$

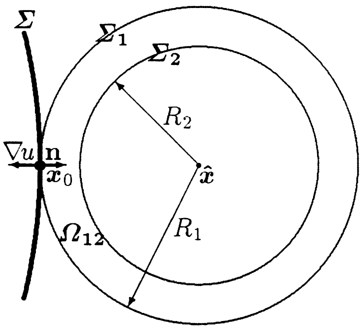

Figure 1. Sketch for the proof of Lemma 3.

Then $v(\mathbf{x}_0) = 0$ and

$$\forall \mathbf{x} \in \Sigma_1 \quad \left(\frac{1}{|\mathbf{x} - \hat{\mathbf{x}}|} - \frac{1}{R_1}\right) = 0 ,$$

$$\forall \mathbf{x} \in \Sigma_2 \quad \left(\frac{1}{|x - \hat{\mathbf{x}}|} - \frac{1}{R_1}\right) = \frac{1}{R_2} - \frac{1}{R_1} .$$

Therefore,

$$\forall \mathbf{x} \in (\Sigma_1 \backslash \mathbf{x}_0) \quad v(\mathbf{x}) > 0 ,$$

$$\forall \mathbf{x} \in \Sigma_2 \quad v(\mathbf{x}) = 1 - u(\mathbf{x}) - \left(\frac{1}{R_2} - \frac{1}{R_1}\right).$$

For $d \ge 2$ one can take $R_1 = 3$ and $R_2 = 3 - 3^{1/d}$, then

$$\forall \mathbf{x} \in \Sigma_2 \quad v(\mathbf{x}) = 1 - u(\mathbf{x}) - \left(\frac{1}{R_2} - \frac{1}{R_1}\right) > 1 - \frac{1}{3 - 3^{1/d}} > 0 .$$

For $2 > d \ge 1$ just take $R_1 = 4$ and $R_2 = 2$, then

$$\forall \mathbf{x} \in \Sigma_2 \quad v(\mathbf{x}) = 1 - u(\mathbf{x}) - \left(\frac{1}{R_2} - \frac{1}{R_1}\right) > 1 - \frac{1}{2^d} - \frac{1}{4} > 0 .$$

Since the harmonic function $v(\mathbf{x})$ is non-negative on the boundary $\Sigma_1 \bigcup \Sigma_2$, it is non-negative (in fact it is positive) in Ω_{12}. Hence $\partial v / \partial \mathbf{n}|_{x_0} \ge 0$. Now,

$$\left.\frac{\partial v}{\partial \mathbf{n}}\right|_{x_0} = \left.\frac{\partial(1 - u(\mathbf{x}))}{\partial \mathbf{n}}\right|_{\mathbf{x}_0} - \left.\frac{\partial}{\partial \mathbf{n}}\left(\frac{1}{|\mathbf{x} - \hat{\mathbf{x}}|} - \frac{1}{R_1}\right)\right|_{\mathbf{x}_0}$$

$$= -\left.\frac{\partial u(\mathbf{x})}{\partial \mathbf{n}}\right|_{\mathbf{x}_0} - \left.\frac{\partial}{\partial \mathbf{n}}\left(\frac{1}{|\mathbf{x} - \hat{\mathbf{x}}|}\right)\right|_{\mathbf{x}_0}$$

$$= -\left.\frac{\partial u(\mathbf{x})}{\partial \mathbf{n}}\right|_{\mathbf{x}_0} - \left(\frac{1}{R_1}\right)^2 \ge 0 .$$

This condition, together with the fact that $\nabla u(\mathbf{x}_0)$ is normal to Σ, provides the inequality

$$|\nabla u(\mathbf{x}_0)| = -\nabla u(\mathbf{x}_0) \cdot \mathbf{n} = -\left.\frac{\partial u(\mathbf{x})}{\partial \mathbf{n}}\right|_{\mathbf{x}_0} \ge \left(\frac{1}{R_1}\right)^2 = \hat{C} .$$

Let us now consider the general case where K and a are arbitrary positive values. Since the harmonic function $u(\mathbf{x})$ is assumed to satisfy Condition 1 with the constant K in the space parametrized by the space variable $\mathbf{x}$, it is straightforward that the harmonic function $u'(\mathbf{x}') = u(\mathbf{x})/a$ satisfies Condition 1 with the constant $K' = 1$ in the space parametrized by the rescaled variable $\mathbf{x}' = (\mathbf{x}/K) a^{1/d}$. Now, since $u(\mathbf{x})$ achieves its maximum value $u(\mathbf{x}_0) = a$ over Σ at $\mathbf{x}_0$, then $u'(\mathbf{x}')$ achieves its maximum value $u'(\mathbf{x}'_0)=u(\mathbf{x}_0)/a=1$ at $\mathbf{x}'_0$ and

$$\nabla' u'(\mathbf{x}')=\frac{1}{a}\,\nabla u(\mathbf{x})\,\frac{K}{a^{1/d}}\,,$$

where ∇' stands for ∇ when referring to the $\mathbf{x}'$ space parameters. The special case then applies to $u'(\mathbf{x}')$ for $K'=1$ and $a'=1$ and it follows that

$$|\nabla u(\mathbf{x}_0)|=a^{1+1/d}K^{-1}\nabla' u'(\mathbf{x}'_0)>\hat{C}a^{1+1/d}K^{-1}\,.$$

This completes the proof of Lemma 3.

We can easily derive the following useful corollary:

Corollary 4 *Let the harmonic potential u satisfy Condition 1 and achieve at* $\mathbf{x}_0\in\Sigma$ *its maximum value* $u(\mathbf{x}_0)=a>0$ *over* $\Omega\bigcup\Sigma$. *Then*

$$a<(K\hat{C}^{-1}|\nabla u(\mathbf{x}_0)\,|)^{d/(d+1)}\,. \tag{1}$$

Now consider two (not necessarily different) potentials f and g belonging to the class $\mathscr{E}(\varepsilon,\delta)$.

Lemma 5 *Let the gradient of the harmonic potential* $h=f-g$ *be normal to the surface* Σ *at the point* $\mathbf{x}_0\in(U_\varepsilon^-\bigcup U_\varepsilon^+)$. *Then*

$$|\nabla h(\mathbf{x}_0)|<\sqrt{\varepsilon^2+2\delta B_{\max}}-\varepsilon\,. \tag{2}$$

Proof. We use an elementary geometric reasoning with reference to Fig. 2 for notations, where 0 stands for $\mathbf{x}_0$. Note the important point that in this figure, A and B both lie on the same side of OC, precisely because $\mathbf{x}_0\in(U_\varepsilon^-\bigcup U_\varepsilon^+)$. This implies that $\mathbf{x}_0$ lies on Σ at a place where the polarity (i.e the sign of $\nabla f\cdot\mathbf{n}$ and $\nabla g\cdot\mathbf{n}$) is well defined. We then have

$$|\vec{AB}|^2+2|\vec{AB}||\vec{BC}|-(|\vec{OA}|-|\vec{OB}|)(|\vec{OA}|+|\vec{OB}|)=0\,. \tag{3}$$

Being positive, $|\vec{AB}|$ is equal to the positive root of eq. (3):

$$|\vec{AB}|=\sqrt{|\vec{BC}|^2+(|\vec{OA}|-|\vec{OB}|)(|\vec{OA}|+|\vec{OB}|)}-|\vec{BC}| \tag{4}$$

$$<\sqrt{|\vec{BC}|^2+2(|\vec{OA}|-|\vec{OB}|)|\vec{OA}|}-|\vec{BC}| \tag{5}$$

$$<\sqrt{\varepsilon^2+2\delta B_{\max}}-\varepsilon\,. \tag{6}$$

The step from (5) to (6) is possible because the function $y(x)=\sqrt{x^2+A}-x$ is decreasing for positive x and A.

4 CONVERGENCE

Let us now consider two functions f and g belonging to $\mathscr{E}(\varepsilon,\delta)$ and achieve the demonstration of convergence by a roundabout way. We will first estimate a bound for $|\nabla f-\nabla g|=|\nabla(f-g)|$ at

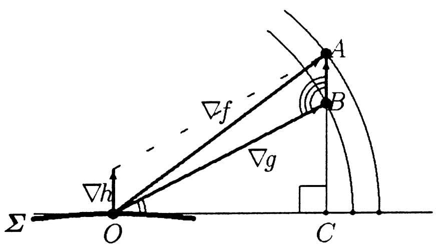

Figure 2. Sketch for the proof of Lemma 5.

the extremum point of $f-g$ on Σ, then use Corollary 4 to get a bound for $|f-g|$ everywhere in $\Omega\bigcup\Sigma$, and finally use Lemma 2 to show that $|\nabla f-\nabla g|\to 0$ everywhere in Ω when $\varepsilon,\delta\to 0$. In fact, the main idea of the proof is this back and forth reasoning between the potential and its gradient on the one hand, and a local (at the extremum) and global (everywhere in Ω) point of view on the other hand.

We first apply some of the results of the preceeding section to the harmonic potential $h=f-g$.

Now the Maximum principle (Courant & Hilbert 1966) says that the function h achieves its minimum and maximum over $\Omega\bigcup\Sigma$ only either at ∞ where $h=0$ (because of Condition 1) or on Σ. If we assume that h is not identically zero, h must have an absolute maximum which cannot be negative. If this maximum is zero, the final result is achieved (see 10 and 11). If it is strictly positive, this maximum must lie at some point $\mathbf{x}_0\in\Sigma$ where $\nabla h(\mathbf{x}_0)$ is normal to the surface Σ.

It is then geometrically obvious that, if $\mathbf{x}_0\in U_\varepsilon^0$, $|\nabla h(\mathbf{x}_0)|\le 2\varepsilon$. If $\mathbf{x}_0\in(U_\varepsilon^-\bigcup U_\varepsilon^+)$, we may apply Lemma 5, and

$$|\nabla h(\mathbf{x}_0)|<\sqrt{\varepsilon^2+2\delta B_{\max}}-\varepsilon\,, \tag{7}$$

so that

$$|\nabla h(\mathbf{x}_0)|<M(\varepsilon,\delta),$$

$$M(\varepsilon,\delta)=\max\,(\sqrt{\varepsilon^2+2\delta B_{\max}}-\varepsilon,\,2\varepsilon)\,. \tag{8}$$

Obviously, when $\varepsilon,\delta\to 0$ the right-hand term of (8) also tends to 0; therefore, inequality (1) of Corollary 4 provides an upper bound for $h(\mathbf{x}_0)$ (provided that we use $\hat{K}=2^{1/d}K$ because condition 1 applies to f and g and not directly to h):

$$h(\mathbf{x}_0)\le(\hat{K}\hat{C}^{-1}M(\varepsilon,\delta))^{d/(d+1)}=C\cdot R(\varepsilon,\delta)\,, \tag{9}$$

where $C=(\hat{K}\hat{C}^{-1})^{d/(d+1)}$ and $R(\varepsilon,\delta)=(M(\varepsilon,\delta))^{d/(d+1)}$.

Taking into account $h(\mathbf{x})|_\Sigma\le h(\mathbf{x}_0)$ and performing a similar reasoning with $h=g-f$, we obtain for sufficiently small ε and δ

$$\sup_{\mathbf{x}\in\Omega}|h(\mathbf{x})|\le\sup_{\mathbf{x}\in\Sigma}|h(\mathbf{x})|\le CR(\varepsilon,\delta)\,. \tag{10}$$

Therefore, $|h(\mathbf{x})|\to 0$ everywhere in $\Omega\bigcup\Sigma$ when $\varepsilon,\delta\to 0$.

By the Weierstrass convergence theorem (Courant & Hilbert 1966), one can then straightforwardly claim that

$$|\nabla f-\nabla g|=|\nabla h|\rightrightarrows 0$$

everywhere in Ω. (Note that Weierstrass' Theorem makes no claim with respect to the behaviour of $|\nabla h|$ on Σ.) Alternatively and more explicitly, one can refer to Lemma 2 and write

$$\iiint_\Omega|\nabla h|^2\,\mathrm{dist}(x,\Sigma)\,dV<D_2C^2R^2(\varepsilon,\delta)\,, \tag{11}$$

which makes the way $|\nabla h|$ gradually reaches 0 when $\varepsilon,\delta\to 0$ more explicit. This completes the proof of Theorem 1.

Inequality (11) makes it explicit that (generally speaking) the convergence is not trivially guaranteed for $|\nabla h(\mathbf{x})|$ when $\mathbf{x}\in\Sigma$ [i.e $\mathrm{dist}(x,\Sigma)=0$]—this is made clear in the Appendix, where a counterexample is constructed. It is in fact only by referring to the additional fact that $h=f-g$, where f and g both satisfy Condition 1, that the convergence in the special case $\mathbf{x}\in\Sigma$ is eventually proven (see the Appendix).

5 PRACTICAL CONSIDERATIONS

The key point for an application of the theoretical bound is the behaviour of the last right-hand term $R(\varepsilon, \delta)$ of inequality (11):

$$R(\varepsilon, \delta) = [\max(\sqrt{\varepsilon^2 + 2\delta B_{\max}} - \varepsilon, 2\varepsilon)]^{d/(d+1)}, \tag{12}$$

which reflects the way that the information we have about the intensity $\|\mathbf{B}\|$ of the field and the location of the equator on Σ translates into some uncertainty concerning the field to be recovered everywhere in Ω.

For practical geomagnetic purposes, we may consider $d = 2$, and rescale ε and δ in units of $B_{\max}$. A constant global $B_{\max}^{2/3}$ factor can then be taken out of $R(\varepsilon, \delta)$, the behaviour of which is then proportional to the right-hand side of (12) when setting $B_{\max} = 1$; the plot is shown in Fig. 3 for ε and δ sufficiently small. The main observation from Fig. 3 is that choosing ε and δ independently small is not the best approach to minimizing $R(\varepsilon, \delta)$ and hence to bounding the *possible* error on a field to be recovered.

If we first consider the case when ε is fixed (i.e the threshold for detecting the polarity of the magnetic field is fixed; recall Condition 2), then $R(\varepsilon, \delta)$ decreases with δ down to some positive lower bound and reaches it when $\delta = 4\varepsilon^2$. Therefore for all $\delta < 4\varepsilon^2$ the *largest possible error for the recovered field* is entirely controlled by the value of ε. Geometrically this reflects the fact that for $\delta < 4\varepsilon^2$ this largest possible error comes from the neighbourhood U_ε^0 of the magnetic equator. This behaviour is reminiscent of the Backus Effect observed in practice when only intensity data are being used to recover the field (see e.g Ultré-Guérard *et al.* 1998). This is because the Backus Effect arises precisely when no knowledge of the equator is being assumed (or equivalently, ε is assumed to be large). We also note here that any additional information concerning B_n in U_ε^0 may then of course significantly improve the bound for the largest possible error. This sheds light on existing empirical results; see e.g. Barraclough & Newitt (1976) and Lowes & Martin (1987).

If we now consider the symmetrical case when δ is fixed, we see that $R(\varepsilon, \delta)$ decreases with ε down to some positive lower bound characterized by $\varepsilon_0 = \sqrt{\delta}/2$ and then starts increasing again for $\varepsilon < \varepsilon_0$. This reflects the fact that for $\varepsilon < \varepsilon_0$ the largest possible error has its origin in $U_\varepsilon^- \bigcup U_\varepsilon^+$. [Because the intersection $(U_\varepsilon^- \bigcup U_\varepsilon^+) \bigcap U_{\varepsilon_0}^0$ is not empty, the bound we derive based on the fact that this region belongs to $U_\varepsilon^- \bigcup U_\varepsilon^+$ is no longer more efficient than the one we derive based on the fact that it also belongs to $U_{\varepsilon_0}^0$.]

Therefore, reducing δ (and increasing the accuracy on the intensity data) constrains the error field behaviour far from the equator but may not bring any constraints globally. By contrast, reducing ε (and locating the equator better) constrains the error field near the equator, but may not bring any constraints on the error field elsewhere.

What *our results* suggest is that the best balance is reached when $\delta = 4\varepsilon^2$. (Note, incidentally, that this result holds independently of the actual values of $B_{\max}$ and d).

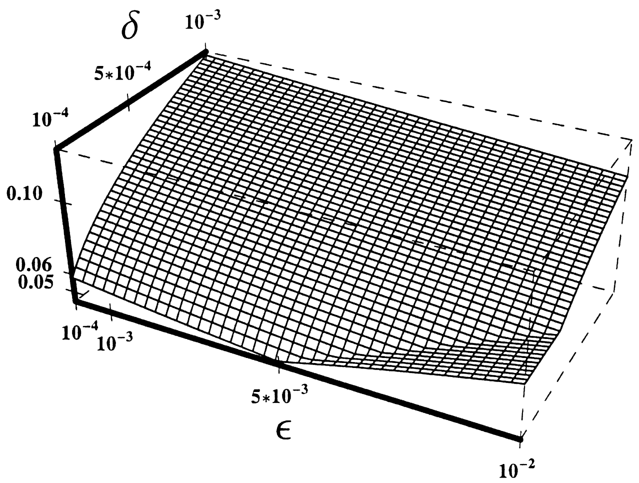

Figure 3. Plot of $R(\epsilon, \delta)$ for $d = 2$ and $B_{\max} = 1$.

6 DISCUSSION AND CONCLUSIONS

The main objective of the paper was to establish and estimate the convergence of the solution computed from an inaccurate set of data $\|\mathbf{B}\|$ on the Earth's surface and an inaccurate location of the dip equator E towards the unique solution of the problem with perfectly accurate data when the error on the data—characterized by the quantities ε and δ—tends to zero. The demonstration itself does not directly produce a practical method to estimate the distance to the true solution, but it does allow us to make some useful statements.

As is known empirically, the best way of making use of intensity data without encountering too large a Backus effect is to carry out additional vectorial measurements in the neighbourhood of the magnetic dip equator; see Lowes & Martin (1987) and Barraclough & Newitt (1976).

According to recent results (Ultré-Guérard *et al.* 1998) (see also our earlier theoretical result; Khokhlov *et al.* 1997), some *a priori* knowledge of where the magnetic equator should lie could instead be used with high efficiency. The present results show that what is important in practice is our ability to identify the neighbourhood $U_{\varepsilon_0}^0$ of the dip equator (out of which the polarity of the field can be assumed to be known). Measuring the vectorial field in the neighbourhood of the dip equator would therefore be most efficient if we focus on the $B_n = \mathbf{B} \cdot \mathbf{n}$ component. The threshold of polarity detection of this component everywhere in this neighbourhood should then only exceed the value $\varepsilon = \sqrt{\delta}/2$, where δ is the relative accuracy with which we are otherwise able to measure the intensity over the surface. The true location of the dip equator need then not be known to a greater accuracy than that required for this ε. The good news about this is that in all practical cases a good accuracy on intensity measurements (i.e a small value of δ) is always much easier to attain than a good accuracy on vectorial measurements (that is, in the present case, a small threshold ε for the detection of the polarity of $B_{\max}$). With a typical value of $\delta = 1$ nT when $B_{\max} = 70\,000$ nT, ε is only required to be of the order of 130 nT. This provides a simple explanation for the success encountered in removing the Backus Effect by including either relatively few vectorial measurements near the dip equator, or relatively weak *a priori* information on the location of this dip equator.

Finally, we note that nothing particular (no special statistical property) was assumed about δ; it could include a regular component such as an uncorrelated external field.

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APPENDIX A: CONVERGENCE AT $\mathbf{x} \in \Sigma$

Let us first stress the fact that knowing that a sequence of harmonic functions $h_k(\mathbf{x})$ converges to zero everywhere in $\Omega \bigcup \Sigma$ does not imply that $|\nabla h_k(\mathbf{x})| \to 0$ for $\mathbf{x} \in \Sigma$. Indeed, assume Σ to be the sphere of unit radius centred at the origin $\mathbf{x}=0$ and consider the harmonic function

$$h_k(\mathbf{x}) = h_k(\rho,\, \theta,\, \varphi) = \frac{P_k(\cos\theta)}{\rho^{k+1}}\,,$$

where $P_k()$ is the Legendre polynomial of degree k and ρ, θ, φ are the spherical coordinates in $\mathbb{R}^3 \backslash 0$. It satisfies (e.g Robin 1957)

$$\iint_\Sigma |h_k|^2 d\sigma = \frac{4\pi}{2k+1} \quad \text{and} \quad \iint_\Sigma |\nabla h_k|^2 d\sigma = 4\pi(k+1)\,.$$

Then, when $k \to 0$, $|h_k| \to 0$ everywhere in $\Omega \bigcup \Sigma$, yet $|\nabla h_k|$ fails to converge (it actually diverges!). The reason for this is that the set $\{h_k\}$ of functions fails to be uniformly bounded in Ω' (just below Σ, as soon as $\rho < 1$, for instance $h_k(\rho,\, 0,\, \varphi) = 1/\rho^{k+1}$ diverges for $k \to \infty$).

In the case of interest in the main text, however, we deal with functions $h_k = f_k - g_k$, where both f_k and g_k satisfy Condition 1. This ensures that $\{h_k\}$ is uniformly bounded in Ω', and the convergence of $|\nabla h_k(\mathbf{x})| \to 0$ at any point $\mathbf{x} \in \Sigma$ can be established by referring to the following 'compactness' statement (see Courant & Hilbert 1966, Chapter IV, Section 2.3).

Proposition 1 *From any uniformly bounded set* $\{h_*\}$ *of regular harmonic functions in some open domain G, a sequence* $\{h_m\}$ *may be selected which converges to a harmonic function uniformly in every closed interior subdomain G′ of G.*

Then, consider $G = \Omega'$ and define the set $\{h_i | i \in \mathscr{I}\}$ as being the set of all possible functions $h = f - g$ as ε, $\delta \to 0$. This set is uniformly bounded by Condition 1. Now if some subset $\{h_j | j \in \mathscr{J}\} \subset \{h_i | i \in \mathscr{I}\}$ fails to satisfy $\nabla h_j(\mathbf{x}) \to 0$ for $\mathbf{x} \in \Sigma$ when ε, $\delta \to 0$ (say $|\nabla h_j(\mathbf{x}_0)| \geq \text{const} > 0$ for all $j \in \mathscr{J}$), Proposition 1 states that a subsequence $\{h_m\}$ of $\{h_j | j \in \mathscr{J}\}$ can nevertheless be extracted which converges uniformly to a harmonic function h_0 in a closed neighbourhood G' of Σ: $\Sigma \subset G' \subset G$. This function h_0 is then analytical in G', but from the proof of Theorem 1 it must be zero in $G' \bigcap \Omega$. The function h_0 must therefore be identically zero in G' and satisfy $\nabla h_0(\mathbf{x}) \equiv 0$ for $\mathbf{x} \in \Sigma$. This contradicts the way the subset $\{h_j | j \in \mathscr{J}\}$ is being defined. Thus all functions of $\{h_i | i \in \mathscr{I}\}$ must satisfy $\nabla h_i(\mathbf{x}) \to 0$ for $\mathbf{x} \in \Sigma$, when ε, $\delta \to 0$.