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SUMMARY

Recovering the internal geomagnetic vector field **B** on and outside the Earth's surface S from the knowledge of only its direction or its intensity $\|\mathbf{B}\|$ on S, and assessing the uniqueness of geomagnetic models computed in this way, have been long-standing questions of interest. In the present paper we address the second problem. Backus (1968, 1970) demonstrated uniqueness in some particular cases, but also produced a theoretical counter-example for which uniqueness could not be guaranteed. Using the same line of reasoning as Backus (1968), we show that adding the knowledge of the location of the dip equator on S to the knowledge of $\|\mathbf{B}\|$ everywhere on S guarantees the uniqueness of the solution, to within a global sign, provided that the dip equator is made of one or possibly several closed curves on S, across which the normal component of the field changes sign (this component not being zero anywhere else).

Key words: Backus Effect, geomagnetic field, inverse problem, planetology.

INTRODUCTION

Recovering the Earth's internal geomagnetic field **B** outside the Earth from measurements that can only be made on the Earth's surface S, or slightly above it, is possible in many ideal circumstances. The magnetic field is derived from a scalar potential U outside S and is completely defined if we have a complete set of measurements of either $\mathbf{B}(S)$ or its horizontal component $\mathbf{B}_H(S)$, or its normal component $B_n(S)$ on S. The determination of **B** (on S and outside S) is then only limited by the accuracy and finite number of measurements (see e.g. Langel 1987); however, in other circumstances some fundamental non-uniqueness exists. A review of the problem was given by Lowes, De Santis & Duka (1995).

Two situations are relevant to geomagnetism. The first, known as the directional problem, arises when only the direction of the field is known on S [note that the inclination $I(S)$ and declination $D(S)$ give both the direction and sense of the field; the corresponding problem is in fact the 'signed directional problem' (see Hulot, Khokhlov & Le Mouël 1997)]. This problem was addressed by several authors, e.g. Kono (1976), Gubbins (1986) and Proctor & Gubbins (1990), but no firm evaluation of the non-uniqueness was reached [except for the 2-D situation studied by Proctor & Gubbins (1990)] until the recent work of Hulot *et al.* (1997). In this paper we pointed out that the space of solutions is linear, showed that the best parameter to look for on S is the number n of loci where the field is known to be either zero (no direction) or normal to the surface S, and proved that the dimension of the space of solutions cannot exceed n in the general case, and $(n-1)$ if the field is known to have no monopole source. Thus an internal geomagnetic field with only two poles (South and North magnetic poles) can be recovered—of course, within a constant multiplier—from directional data gathered at the Earth's surface.

The second situation is the one for which only the intensity $B(S)$ is known on S. Backus (1970) showed, with the help of an *ad hoc* counter-example, that uniqueness of the recovered field (within, of course, a ± 1 factor) could not in general be guaranteed. However, Backus (1968) also pointed out that there exist situations for which uniqueness can be guaranteed, such as when the field is *a priori* known to be either a finite sum of spherical harmonics or never tangent to S. The exact circumstances leading to non-uniqueness, the extent of this non-uniqueness and the status of the geomagnetic field with respect to this problem remain to be assessed. What is certain is that in practice some large errors arise, especially in the neighbourhood of the magnetic equator (e.g. Stern & Bredekamp 1975; Ultré-Guérard 1996). This is often referred to as the *Backus Effect* [although the actual errors arise as the result of two effects: the finite number of data and the fundamental non-uniqueness pointed out by Backus (Lowes 1975)]. Some efforts have been devoted to reducing this error by adding a possibly small, but well-chosen set of vectorial measurements (Hurwitz & Knapp 1974; Stern & Bredekamp 1975; Barraclough & Nevitt 1976; Lowes & Martin 1987; Ultré-Guérard 1996). A useful empirical result that has been known for some time is that errors can be reduced when the vectorial measurements come from near the equator

(Barraclough & Nevitt 1976; Lowes & Martin 1987). Adding even low-quality vectorial data to the good intensity data greatly reduces the Backus Effect (Holme & Bloxham 1995). The purpose of the present paper is to provide a theoretical basis for this result.

UNIQUENESS THEOREM

Basic assumptions. In this section we assume that all curves and surfaces are *smooth*, which means that they are differential submanifolds in R^3; all functions, defined in the unbounded part of the space, have a zero limit at ∞. We use the notation ∂D for the smooth boundary of an open domain, D.

Let S denote a closed bounded surface in R^3, U denote a harmonic function defined on and outside S, and $\mathbf{B}=\nabla U$. Backus (1970) stated that the surface intensity data $\|\mathbf{B}\||_S$ alone is not enough to recover $\mathbf{B}$ even up to the sign. In other words, there exist pairs, say U and $\tilde{U}$, such that the corresponding gradients $\mathbf{B}=\nabla U$ and $\tilde{\mathbf{B}}=\nabla\tilde{U}$ satisfy

$$\|\mathbf{B}\||_S=\|\tilde{\mathbf{B}}\||_S \quad \text{and} \quad \mathbf{B}\neq\pm\tilde{\mathbf{B}}.$$

Still assuming that $\|\mathbf{B}\||_S=\|\tilde{\mathbf{B}}\||_S$, let us now consider the sum $V=\tilde{U}+U$ and the difference $W=\tilde{U}-U$. Obviously,

$$\nabla V\cdot\nabla W|_S=\|\tilde{\mathbf{B}}\|^2|_S-\|\mathbf{B}\|^2|_S\equiv 0\,. \tag{1}$$

By the Maximum Principle (Kellogg 1953), the harmonic function W achieves its minimum and maximum values only at ∞ or on S. However, the following is also known (Bers, John & Schechter 1964, pp. 151–152, Theorem III; Hulot *et al.* 1997).

Lemma. *Let W be a function different from a constant, harmonic in an open domain D, continuous together with its first-order partial derivatives up to the differentiable boundary $S=\partial D$ and assuming a maximum (minimum) at some point lying on S. At this point one then has $\nabla W=\alpha\mathbf{n}$ for some $\alpha\neq 0$, where $\mathbf{n}$ denotes the normal to S.*

From this Lemma we see that the vector ∇W is non-zero and normal to S at the extrema of W lying on S. Taking into account (1), we may then conclude that at such extrema of W the vector field ∇V must be tangent (in a general sense) to S.

Now let us assume the following.

Additional assumption. Both U and $\tilde{U}$ have no monopole sources and share the same dip equator, made of one or several closed connected curves L_E across which both $\nabla U\cdot\mathbf{n}$ and $\nabla\tilde{U}\cdot\mathbf{n}$ change sign, $\nabla U\cdot\mathbf{n}$ and $\nabla\tilde{U}\cdot\mathbf{n}$ being zero only on L_E.

This implies that the equator L_E divides S into regions where $\nabla U\cdot\mathbf{n}$ and $\nabla\tilde{U}\cdot\mathbf{n}$ are never zero and are either always of a common sign (i.e. of the same polarity) or always of opposite signs (i.e. of opposite polarities). Let us first consider the case when U and $\tilde{U}$ are of the same polarity. Then, $\nabla V=\nabla\tilde{U}+\nabla U$ has the curve L_E as the only set of tangency points to S (we add two vector fields which are similarly oriented with respect to the normal $\mathbf{n}$), and the extrema of W on S must lie on L_E. However, on L_E, $\nabla W=\nabla\tilde{U}-\nabla U$ is also tangent to S. It follows that at any extremum of W on S, ∇W must be non-zero and both normal and tangent to S, which is impossible. As a result, W can achieve its extrema only at infinity, where it must be zero. Thus, $W=\tilde{U}-U$ is zero everywhere and $\tilde{U}=U$. If we next consider the case when U and $\tilde{U}$ are of opposite polarities, the same reasoning can apply, provided that one changes U into $-U$. In that case the conclusion is that $\tilde{U}=-U$. We may thus claim the following.

Theorem *In the above assumptions on the harmonic potentials U and $\tilde{U}$ the two gradient fields $\mathbf{B}=\nabla U$ and $\mathbf{B}=\nabla\tilde{U}$ are either equal everywhere or opposite everywhere if, on S, they have the same intensity $\|\mathbf{B}\|=\|\tilde{\mathbf{B}}\|$ and share the same dip equator, L_E.*

In other words, knowing the intensity and the location of the dip equator on the surface S guarantees the uniqueness of the field to within a global sign. This theorem provides a generalization of the theorem by Backus (1968) (Theorem III, p. 212), stating that if $\nabla U\cdot\mathbf{n}>0$ everywhere on S (in which case, from our point of view, L_E is empty), the field is uniquely determined to within a global sign from the knowledge of $\|\nabla U\|$ on S.

DISCUSSION AND CONCLUSIONS

Our result is not of straightforward application: measuring $\|\mathbf{B}\|$ on S does not provide the location of the dip equator L_E where $\mathbf{B}$ is horizontal. However, as collecting additional vectorial data in the vicinity of the equator obviously puts strong constraints on the location of the dip equator, it now becomes clear why such data have been successful in improving the possibility of determining the whole field in a unique way. In a follow-up paper, 'On the Backus Effect—II', we will address the question of the errors and loss of uniqueness due to a possible mislocation of L_E. A curve L'_E close to the true dip equator can be obtained by correcting for secular variation the dip equator of a vectorial model for an earlier epoch.

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REFERENCES

Backus, G., 1968. Application of a non-linear boundary value problem for Laplace's equation to gravity and geomagnetic intensity surveys, *Q. J. Mech. Appl. Math.*, **21**, 195–221.

Backus, G., 1970. Non-Uniqueness of the external geomagnetic field determined by surface intensity measurements, *J. geophys. Res.*, **75**, 6339–6341.

Barraclough, D.R. & Nevitt, C.E., 1976. The effect of observational errors on geomagnetic field models based solely on total-intensity measurements, *Phys. Earth planet. Inter.*, **13**, 123–131.

Bers, L., John, F. & Schechter, M., 1964. *Partial Differential Equations*, Wiley, New York, NY.

Gubbins, D., 1986. Global models of the magnetic field in historical times: augmenting declination observations with archeo- and paleomagnetic data, *J. Geomagn. Geoelectr.*, **38**, 715–720.

Holme, R. & Bloxham, J., 1995. Alleviation of the Backus effect in geomagnetic field modelling, *Geophys. Res. Lett.*, **22**, 1641–1644.

Hulot, G., Khokhlov, A. & Le Mouël, J.L., 1997. Uniqueness of mainly dipolar magnetic field recovered from directional data, *Geophys. J. Int.*, **129**, 347–354.

Hurwitz, L. & Knapp, D.G., 1974. Inherent vector discrepencies in geomagnetic main field models based on scalar F, *J. geophys. Res.*, **79**, 3009–3013.

Kellogg, O.D., 1953. *Foundations of Potential Theory,* Dover, New York, NY.

Kono, M., 1976. Uniqueness problems in the spherical harmonic analysis of the geomagnetic field direction data, *J. Geomagn. Geoelectr.,* **28,** 11–29.

Langel, R.A., 1987. The main field, in *Geomagnetism,* Vol. 1, pp. 249–512, ed. Jacobs, J.A., Academic Press, London.

Lowes, F.J., 1975. Vector errors in spherical harmonic analysis of scalar data, *Geophys. J. R. astr. Soc.,* **42,** 637–651.

Lowes J.F. & Martin, J.E., 1987. Optimum use of satellite intensity and vector data in modelling the main geomagnetic field, *Phys. Earth planet. Inter.,* **48,** 183–192.

Lowes F.J., De Santis, A. & Duka B., 1995. A discussion of the uniqueness of a Laplacian potential when given only partial field information on a sphere, *Geophys. J. Int.,* **121,** 579–584.

Proctor M.R.E. & Gubbins, D., 1990. Analysis of geomagnetic directional data, *Geophys. J. Int.,* **100,** 69–77.

Stern, D.P. & Bredekamp, J.H., 1975. Error enhancement in geomagnetic models derived from scalar data, *J. geophys. Res.,* **80,** 1776–1782.

Ultré-Guérard, P., 1996. Du paléomagnétisme au géomagnétisme spatial: analyse de quelques séquences temporelles du champ magnétique terrestre, *Thése,* Institut de Physique du Globe de Paris.