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Electron scattering and nonlinear trapping by oblique whistler waves: The critical wave intensity for nonlinear effects

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In this paper, we consider high-energy electron scattering and nonlinear trapping by oblique whistler waves via the Landau resonance. We use recent spacecraft observations in the radiation belts to construct the whistler wave model. The main purpose of the paper is to provide an estimate of the critical wave amplitude for which the nonlinear wave-particle resonant interaction becomes more important than particle scattering. To this aim, we derive an analytical expression describing the particle scattering by large amplitude whistler waves and compare the corresponding effect with the nonlinear particle acceleration due to trapping. The latter is much more rare but the corresponding change of energy is substantially larger than energy jumps due to scattering. We show that for reasonable wave amplitudes ~10–100 mV/m of strong whistlers, the nonlinear effects are more important than the linear and nonlinear scattering for electrons with energies ~10–50 keV. We test the dependencies of the critical wave amplitude on system parameters (background plasma density, wave frequency, etc.). We discuss the role of obtained results for the theoretical description of the nonlinear wave amplification in radiation belts. © 2014 AIP Publishing LLC.

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I. INTRODUCTION

The resonant interaction of electrons with whistler-mode waves plays an important role in electron scattering and acceleration in many plasma systems: radiation belts, solar wind, shock waves, and planetary magnetotails. The general approach for the description of such an interaction is based on the quasi-linear theory of charged particle scattering by uncorrelated small amplitude waves. The alternative approach corresponding to the consideration of the wave-particle nonlinear interaction is generally applied for systems with high enough wave amplitudes. This approach is based on the analysis of nonlinear equations of the charged particle motion. The dynamical system approach can also be applied for analysis of such systems. Almost all the modern global three-dimensional models describing the evolution of an ensemble of charged particles resonantly interacting with whistler waves take into account only quasi-linear effects of scattering and do not include any nonlinear effects (see review in Ref. 45 and references therein).

Recently, spacecraft observations in the Earth radiation belts at the bow-shock, and in the magnetotail suggest that a significant part of the whistler wave population actually consists of high-amplitude waves. The electron resonant interaction with such waves can have a nonlinear character, including effects of electron trapping. Thus, the possible role of these nonlinear effects for large electron ensembles is at this time an open and pressing question.

The character of the wave-particle resonant interaction is determined by the competition of two factors: wave intensity and inhomogeneity of the background magnetic field. For strong enough wave amplitude, the resonant interaction is nonlinear with a possible particle trapping. The threshold on the wave amplitude necessary for particle trapping was found for several systems with various wave-modes. However, the possibility of particle trapping and its subsequent nonlinear acceleration does not necessary means that this process plays an important role in realistic systems. In fact, particle trapping is a probabilistic process, i.e., only a small portion of resonant particles can be trapped by a wave during the first resonant interaction, while other resonant particles should be scattered. A probability of trapping (ratio of trapped particles to the total amount of resonant particles) can be defined for each particular system. For relativistic electron interaction with oblique whistler waves, the probability of trapping was calculated and numerically tested for both Landau and cyclotron resonances (see Refs. 5 and 6). Thus, these probabilities can be used to estimate the relative impact of nonlinear trapping in particle acceleration. This impact can be either small (when particles are so rarely trapped that the much more frequent scattering is globally more effective at changing particle energy) or large (when the change of particle energy due to trapping is so large as compared to changes due to scattering that even a small probability of trapping results in a more effective nonlinear acceleration). Moreover, in some particular situations a general balance could be reached between scattering and trapping, with both processes playing comparable roles in particle acceleration/deceleration. In the following, we address the preceding questions in the case of...
electron resonant interaction with oblique whistler-mode waves. First, we derive the relevant equations describing nonlinear electron scattering and trapping, which then allows us to compare in details these two processes.

II. MAIN EQUATIONS

We consider the interaction of relativistic electrons (with rest mass $m_0$ and charge—$e$) with a strong oblique whistler wave described by the scalar potential $\Phi = \Phi_0 \sin \phi$ with phase $\phi$. We focus on the Landau resonance and, as a result, can assume the conservation of the magnetic moment of particles (i.e., the corresponding wave phase does not depend on the coordinate transverse to the background magnetic field). The model of the magnetic field is chosen so that after averaging over electron gyrorotation, the magnetic field magnitude $B(z)$ depends only on the coordinate $z$ along field lines (i.e., this magnetic field model does not include any effect of curvature of field lines). In this case, the Hamiltonian system describing the Landau resonant interaction of an electron and quasi-electrostatic wave can be written as

$$H = \gamma - ea_0(z) \sin \phi,$$

$$\gamma = \sqrt{1 + p_z^2 + \xi b(z)},$$

$$\phi = \phi_0 + \int k_0 |z'| dz' - cot.$$

(1)

In system (1), we use dimensionless variables: particle parallel momentum $p_z$ is normalized on $m_0 c$; the coordinate $z$ is normalized on $R_0$ (here, $R_0 = R_E L$ is the scale of the background magnetic field inhomogeneity, $L$ denotes the L-shell, while $R_E$ is the Earth radius); the background magnetic field is normalized to its equatorial value $b(z) = B(z)/B_0$; the dimensionless magnetic moment is $\xi = (\gamma_0^2 - 1) \sin^2 q_0$, where $\gamma_0$ and $q_0$ are initial electron gamma factor and equatorial pitch-angle; wavenumber is normalized as $(k_0, k_L) \rightarrow (k_0, k_L)/R_0$; the normalized wave frequency is $\omega = \omega_{R_E} c = \omega_{m_0 L}$, where $\omega_{m_0} = \omega m_0 c/e B_0$ and $\gamma = e B_0 R_E m_0 c^2$; the wave amplitude is normalized as $e \Phi_0 / m_0 c^2 = u_0(q)$, where $u_0(q) = u(q)J_0(q)$ and function $u(q) \in [0, 1]$ describes the distribution of the wave amplitude along field lines (this distribution is derived from the statistics of spacecraft observations, see details in Ref. 5).

While $J_0(q)$ is the Bessel function of the first order with the argument $q = (k_0/L) \sqrt{\xi b(z)}$; and $\phi_0$ is the initial wave phase. In contrast to several previous studies,5,15,49 we consider the wave-particle interaction for a coherent monochromatic wave occupying the entire flux tube with a given amplitude profile. This simplification allows us to reduce a number of free parameters. Effects of a finite-length of a wave packet should be considered further.

The pair of conjugate variables in Eq. (1) is $(z, p_z)$. Thus, the corresponding Hamiltonian equations have a form

$$\dot{p}_z = -\frac{1}{2} \xi b + c k_0 u_0 \cos \phi,$$

$$\dot{z} = p_z / \gamma,$$

(2)

where $' = d/dz$. To consider the resonant wave-particle interaction, we rewrite Eq. (2) in terms of wave-phase along the particle trajectory

$$\dot{\phi} = \frac{\dot{p}_z}{\gamma} - \omega,$$

$$\gamma = \sqrt{1 + \xi b} / \sqrt{1 - (\dot{\phi} + \omega)^2 / k_0^2}.$$  

(3)

We take one additional derivative of the first equation from system (3) to obtain the equation for $\dot{\phi}$,

$$\ddot{\phi} = \frac{k_0 |p_z|}{\gamma} \frac{\ddot{z}}{\gamma} - k_0 \dot{p}_z / \gamma.$$  

(4)

We substitute Eqs. (2) and (3) into Eq. (4) to get

$$\frac{\ddot{\phi}}{1 - (\dot{\phi} + \omega)^2 / k_0^2} = k_0 \left( \frac{\dot{\phi} + \omega}{k_0^2} + \frac{k_0 \dot{p}_z}{\gamma} \right) - \frac{1}{2} \frac{\ddot{z}}{b(z)} \frac{(\dot{\phi} + \omega)^2}{k_0} - \frac{1}{1 - (\dot{\phi} + \omega)^2 / k_0^2} \frac{k_0^2 (\dot{\phi} + \omega)^2}{k_0^2}.$$  

(5)

where $\dot{p}_z$ should be substituted from Eq. (2). Considering the above equation in the vicinity of the Landau resonance $\dot{\phi} = 0$ gives

$$\frac{\ddot{\phi}}{k_0} = k_0 \dot{\phi}^2 - \frac{\ddot{\phi}}{b(z)} + \frac{k_0 e u_0}{2 \gamma_0^2} \cos \phi,$$

(6)

where $v_R(z) = \omega / k_0 \gamma$ and

$$\gamma_R = 1 / \sqrt{1 - v_R^2},$$

$$\gamma = \gamma_R \sqrt{1 + \xi b}.$$  

(7)

In Eq. (6), the phase $\phi$ changes much faster than the position on the $z$-axis: $\dot{\phi} \gg \gamma_R$. Thus, following to Ref. 23 we can consider Eq. (6) as an equation for $\phi$ with $z$ as a slowly varying parameter. In this case, Eq. (6) can be rewritten in a form

$$\dot{\phi} = \frac{\gamma_R^2}{2 \gamma^2 k_0} \left( \ddot{\phi} - 2 \frac{\ddot{z}}{b(z)} \frac{k_0^2}{k_0^2} + c u_0 \cos \phi,$$

(8)

where we introduce $P = \gamma_R^2 k_0 \ddot{\phi} / k_0^2$. It can be shown that $P$ and $\phi$ are conjugate variables for the Hamiltonian $H_\phi$ (see Appendix A in Ref. 6)

$$H_\phi = \frac{1}{2} \frac{k_0^2}{\gamma_R^2} P^2 + D(z) \phi - e u_0 \sin \phi,$$

(9)

where

$$D(z) = \frac{\gamma_R^2}{2 \gamma^2 k_0} \left( \ddot{\phi} - 2 \frac{\ddot{z}}{b(z)} \frac{k_0^2}{k_0^2} \right),$$

$$U_\phi(z) = D(z) \phi - e u_0 (z) \sin \phi.$$  

(10)
The effective potential energy $U_{\phi}$ of Hamiltonian (9) depends via the $z$ coordinate on both the wave intensity variation along field line and the magnetic field inhomogeneity. Phase portraits of Hamiltonian (9) for three cases $D > \varepsilon u_0$, $D = \varepsilon u_0$, and $D < \varepsilon u_0$ are displayed in Fig. 1. The presence of a region with closed trajectories in the case with $D < \varepsilon u_0$ corresponds to a possible particle trapping. This effect is considered in Sec. IV. However, trapping is a probabilistic process—i.e., each passage through resonance does not result in trapping. If trapping is not realized, the particle is scattered at resonance with the wave. Moreover, for the system with $D > \varepsilon u_0$, only scattering is possible. We treat this effect in Sec. III.

III. SCATTERING

In this section, we consider the evolution of particle energy in the vicinity of resonance in the case when a particle is not trapped by the wave. Particle momentum $P$ is only slightly changed by scattering at the resonance crossing. This change can be found from Eq. (8)

$$\Delta P = \int_{-\infty}^{\infty} \dot{P} dt = 2\varepsilon u_0(z) \int_{-\infty}^{\infty} \cos \phi d\phi d\phi,$$  \hspace{1cm} (11)

where $\tau$ and $\phi$ are time moment and phase at resonance. We use $\dot{\phi} = Pk^2/\gamma_R^2$ and Eq. (9) to rewrite Eq. (11) as

$$\Delta P = \varepsilon u_0 \sqrt{\frac{2\gamma_R^2}{k^2}} \int_{-\infty}^{\infty} \cos \phi d\phi d\phi.$$

The resonant phase $\phi^*$ is defined from the equation $\dot{\phi} = 0$. The particle Hamiltonian $H_\phi$ can be written at resonance ($\phi = \phi^*, P = 0$). Thus, we can rewrite Eq. (12) in a form

$$\Delta P = \varepsilon u_0 \sqrt{\frac{2\gamma_R^2}{k^2}} \int_{-\infty}^{\phi^*} \cos \phi d\phi d\phi$$

$$= \varepsilon u_0 \sqrt{\frac{2\gamma_R^2}{k^2} D} \int_{-\infty}^{\phi^*} \cos \phi d\phi d\phi$$

$$= \varepsilon u_0 \sqrt{\frac{2\gamma_R^2}{k^2} D} f(\theta, a),$$

where $a = \varepsilon u_0/D$, $\theta = (\phi^* - a \sin \phi^*)/2\pi$, and $\phi^*$ is defined by the equation $2\pi \theta - \phi^* + a \sin \phi^* = 0$. The function $f(\theta, a)$ is shown in Fig. 2 for different values of $a$. One can see that $f$ is a periodic function of $\theta$. The value of $\theta$ is determined by the exact value of the fast oscillating phase $\phi$ at resonance. Thus, $\theta$ can be assumed to be a random variable with uniform distribution over $\theta \in [0, 1]$ (see Refs. 4, 32, and 33). In this case, we can consider the average value of $\Delta P$ and the corresponding dispersion $\text{Var}(\Delta P)$ around this average

$$\langle \Delta P \rangle = \varepsilon u_0 \sqrt{\frac{2\gamma_R^2}{k^2} D} \langle f(\theta, a) \rangle_0 = P_0 \langle f \rangle_0,$$

$$\text{Var}(\Delta P) = \langle \Delta P^2 \rangle - \langle \Delta P \rangle^2 = P_0^2 \langle (f^2) \rangle_0 - \langle f \rangle_0^2.$$

Both terms $\Delta P$ and $\text{Var}(\Delta P)$ are shown in Fig. 3 as functions of $a$. For $a < 1$ we have $\langle P \rangle = 0$, while for $a > 1$ there is a finite regular drift of $P$ (i.e., $\langle P \rangle \neq 0$). Since $P$ is a function of $\gamma$, the preceding expressions for $\langle P \rangle$ and $\text{Var}(\Delta P)$ can be rewritten under the form of corresponding expressions for

![FIG. 1. Phase portraits of system (9).](image1)

![FIG. 2. Profiles of $f(\theta, a)$ function for four values of $a$.](image2)

![FIG. 3. $\Delta P$ and $\text{Var}(\Delta P)$ as functions of $a$.](image3)
the mean and variance of the particle energy ($\gamma$). To this aim, we rewrite $\Delta P$ in a form

$$\Delta P = \frac{\gamma^2}{k_{||}} \Delta \phi = \frac{\gamma^2}{k_{||}} \left( k_{||} \Delta p_z - k_{||} \gamma \frac{\partial}{\partial \gamma} \right).$$  

(15)

We can use $\Delta n = p_z \Delta p_z$ and $p_z \gamma = v_R$ to rewrite Eq. (15) as

$$\Delta n = v_R k_{||} \Delta P.$$  

(16)

Thus, we can substitute Eq. (16) into Eq. (14) to finally get the relevant expressions for mean energy change and variance

$$\langle \Delta n \rangle = \gamma_0 (f)_0,$nolipagebreak

(17)

$$\text{Var}(\Delta n) = \gamma_0 \left( \langle f \rangle^2_0 - \langle f \rangle^2_0 \right),$$

where all variables (except for the dimensionless magnetic moment $\zeta$) must be calculated at the position of resonance $z = z_R$, so that $\gamma_0$ is a function of $z_R$.

To demonstrate the effect of charged particle scattering at the resonance, we consider the particular case of a wave propagating at the Gendrin angle: $k_0 = k_0 b(z)$; with $k_0 = 2\alpha_{pe}/\omega_0$, and $\alpha_{pe}$ is the ratio of plasma frequency and electron gyrofrequency at the equator (see details on this approximation for $k_{||}$ in Ref. 6). For the Earth radiation belts $\alpha_{pe}$ is a function of $L$-shell. In this case, Eq. (17) takes a form

$$\gamma_0 = \frac{1}{\sqrt{k_0 k_R}} \frac{2v_R^2 - 1}{\sqrt{\zeta^2 b + 2j^2 v_R^2}},$$

$$a = \frac{v_R^2}{D} = \frac{1}{\sqrt{\zeta^2 b + 2j^2 v_R^2}}.$$  

(18)

Here, factor $k_0 k_R = eE_{||} R_0 / m_e c^2$ is about one for high-amplitude waves ($E_{||0} = \Phi_{0} b_k$ is the electric field amplitude), while factor $1/\sqrt{k_0} \ll 1$ determines the smallness of the energy change for a single passage through the resonance. The position of the resonance $z_R$ is determined by the equation $\gamma_0 = \gamma_R (z_R) \sqrt{1 + \zeta^2 b (z_R)}$, where $\gamma_0$ is the initial particle energy. Thus, the resonance location $z_R$ depends on $\gamma_0$ and equatorial pitch-angle $\theta_0$. For various initial energies ($\gamma_0$), we have determined $z_R (\gamma_0)$ and plotted $\gamma_0$ as a function of $\theta_0$ in Fig. 4, using the same wave and plasma parameters as in Ref. 5. The decrease of the particle energy corresponds to a shrinking range of $\theta_0$ where $\gamma_0 \neq 0$. The absolute value of $\gamma_0$ is about $k_0 e / \sqrt{k_0}$. The mean energy change is proportional to $\gamma_0 \sim u_0 k_0 b_0 / \sqrt{k_0}$ at small equatorial pitch-angle $\theta_0 < 45^\circ$ where it becomes independent of $\theta_0$. There it increases with decreasing energy in Fig. 4, due to a general increase of wave intensity $u_0 (z_R)$ as latitude of resonance is reduced for a realistic latitudinal distribution of oblique wave intensity based on satellite measurements (see Ref. 5).

To further examine the effect of scattering, we numerically integrate Eq. (2) for two trajectories. The first trajectory corresponds to initial $\theta_0 = 10^\circ$. In this case, the averaged $\langle \Delta n \rangle$ is equal to zero and we should obtain only random jumps of $\gamma$ with the average amplitude $\sim \sqrt{\text{Var}(\Delta n)}$. This trajectory is displayed in the top panels of Fig. 5. Particle oscillations between mirror points correspond to a closed trajectory in the $(z, p_z)$ plane. At the resonance $z_R \approx 0.625$, the particle experiences jumps in energy and scattering of $z$ and $p_z$ values. The corresponding energy jumps are randomly negative or positive, with an average jump amplitude close to the theoretical prediction (see the right top panel in Fig. 5).

The second trajectory (bottom panels in Fig. 5) is integrated with an initial $\theta_0 = 60^\circ$. In this case, the variance $\sim \sqrt{\text{Var}(\Delta n)}$ is substantially smaller than the mean $\langle \Delta n \rangle$. Thus, at each passage through resonance the scattering of the particle should correspond to a decrease of its energy due to $\langle \Delta n \rangle < 0$. Indeed, one can see in the right bottom panel in Fig. 5 this expected behavior (decrease) of $\gamma$.

Although the integration of individual particle trajectories already shows a good agreement between numerical results and analytical estimates, a more comprehensive check of Eq. (17) requires to consider a large particle ensemble. In this case, random fluctuations can be averaged and mean values of $\text{Var}(\Delta n)$, $\langle \Delta n \rangle$ can be obtained numerically as functions of the initial energy and pitch-angle. Results of such massive tests are shown in Fig. 6 for three energies and two values of the wave-amplitude. Each point (symbol) in Fig. 6 corresponds to an averaged value obtained by integration of $10^4$ particle trajectories. One can see that all dependencies (resonant latitude $\gamma$ corresponding to $z_R$, $\text{Var}(\Delta n)$, and $\langle \Delta n \rangle$) on the initial particle pitch-angle are well reproduced. It demonstrates that the analytical approximations given by Eq. (17) can be safely used to describe the behavior of large particle ensembles.

**IV. TRAPPING**

Particle trapping by quasi-electrostatic whistler-mode waves into the Landau resonance was described in details in Refs. 5 and 6. Here, we reproduce the main results, to compare the efficiency of trapping and scattering processes. An example of particle trajectory with trapping of particle by the
wave is shown in Fig. 7. This trajectory is obtained by numerical integration of system (1) with the same parameters as used for the trajectory shown in Fig. 5 (bottom panels), but with a two times larger wave amplitude. Initially, the particle oscillates between mirror points (bounce oscillations) and after a certain time, it becomes trapped by the wave. The trapped particle moves with the wave to higher latitude and is strongly accelerated. One trapping-escape event results in an energy gain about 70 keV in accordance with previous estimates.5,6,39

In contrast to scattering, the probability of particle trapping is small: only some limited portion of particles passing through the resonance becomes trapped.4,33 The ratio of the number of trapped particles to the total number of particles passed through the resonance can be called a probability of trapping \( P \). An analytical expression for \( P \) has been derived in Ref. 6 and tested numerically in Ref. 7. Over a realistic parameter range, \( P \) is defined by the expression

\[
P = \frac{\sqrt{\epsilon}}{4\pi\omega_{pe}k_D} \frac{\partial S}{\partial z},
\]

where the area \( S \) is shown by grey color in Fig. 1 and is defined by the equation

\[
S = 2^{2/3} \gamma R b \sqrt{m_0} \int_{\phi_a}^{\phi_b} \sqrt{\frac{1}{a} (\phi_1 - \phi) - \sin \phi_1 + \sin \phi} d\phi,
\]

where \( \phi_a = -\arccos(1/a) \), \( \phi_a \) is a root of the equation \( \phi_1 - \phi + a \sin \phi - a \sin \phi_1 = 0 \) different from \( \phi_s \) (see details in Appendix A of Ref. 6).

V. SCATTERING VS. TRAPPING

For given initial particle pitch-angle \( \alpha_0 \) and energy \( \gamma_0 \), we can compare estimates of energy jumps due to scattering \( \Delta E_{\text{scat}} \) and due to trapping \( \Delta E_{\text{trap}} \):

\[
\Delta E_{\text{scat}} = m_e c^2 (1 - \Pi) \sqrt{(\Delta \gamma)^2 + \text{Var}(\Delta \gamma)},
\]

\[
\Delta E_{\text{trap}} = m_e c^2 \Pi \Delta \gamma_{\text{trap}},
\]

where the expression for the energy jumps due to trapping \( \Delta \gamma_{\text{trap}} \) was derived in Ref. 6. The change \( \Delta \gamma_{\text{trap}} \) can be defined as a difference of the particle gamma factors at the point of the escape from the resonance and initial value of \( \gamma \). The ratio \( \Delta E_{\text{trap}}/\Delta E_{\text{scat}} \) is shown in Fig. 8 for different particle energies and wave amplitudes. One can see that \( \Delta E_{\text{trap}}/\Delta E_{\text{scat}} \) profiles look similar with \( \Pi \) profiles, i.e., the main variation with \( \alpha_0 \) is provided by the probability variation with \( \alpha_0 \). For small energies \( \sim 10 \text{ keV} \), almost all the \( \alpha_0 \)-range available for resonant interaction corresponds to a predominance of acceleration due to trapping. For larger electron energies, the available \( \alpha_0 \)-range for resonance becomes shorter and only half of this range corresponds to \( \Delta E_{\text{trap}}/\Delta E_{\text{scat}} > 1 \). However, an increase of wave amplitude results
in a widening of the $\alpha_0$-range corresponding to $\Delta E_{\text{trap}} / \Delta E_{\text{scat}} > 1$. This effect is due to the increase of $\Delta E_{\text{trap}}$ with wave amplitude $\varepsilon$ (see Ref. 6), while the ratio $\Pi/\gamma_0$ decreases with $\varepsilon$ as $\Pi/\gamma_0 \sim \sqrt{\varepsilon} / \varepsilon \sim 1/\sqrt{\varepsilon}$.

For a given equatorial pitch-angle $\alpha_0$ and energy $\gamma_0$, one can further determine the critical value of the wave amplitude $E_{\|0}$ such that changes in energy due to trapping exceed scattering-induced changes, i.e., such that $\Delta E_{\text{trap}} > \Delta E_{\text{scat}}$ for $E_{\|0} > E_{\|0}^*$. For several values of energy, the profiles $E_{\|0}^*(\alpha_0)$ are shown in Fig. 9. First, we note that for any electron energy below 80 keV, there exists an $\alpha_0$-domain such that the corresponding value of $E_{\|0}^*$ is rather small ($< 25$ mV/m). Waves with substantially larger amplitudes have often been observed in the radiation belts.\textsuperscript{1,12,13} However, the pitch-angle range where $E_{\|0}^* < 100$ mV/m is really large (with a domain $0 \leq \alpha_0 \leq 45^\circ$) only for small particle energies $\sim 10$ keV. For larger energies, the range of $\alpha_0$ where $E_{\|0}^* < 100$ mV/m is only about $5^\circ$–$10^\circ$.
A comparison of the right and left panels in Fig. 9 shows that the value of the normalized wave frequency does not influence significantly the relationship between nonlinear acceleration and scattering levels. This absence of effect can be explained by the fact that the wave phase velocity \( v_R = \omega/k \) is independent of the wave frequency for waves propagating at (or near to) the Gendrin angle \( (k \sim \omega) \). For much more oblique waves propagating close to the resonance-cone angle \( (k \sim \omega^2) \), see Ref. 1, we expect a stronger dependence of all system parameters (including \( E_{\text{in}} \)) on the wave frequency.

**VI. DISCUSSION AND CONCLUSIONS**

In this paper, we consider electron resonant interaction with high-amplitude oblique whistler waves propagating in an inhomogeneous magnetic field. More specifically, we compare two different effects of wave-particle interaction: scattering and nonlinear trapping. The latter one can lead to a very strong acceleration of individual particles, but the corresponding probability of trapping is small. As a result, a weak energy scattering of particles by the waves may be more effective than acceleration due to trapping even for high-amplitude waves. Thus, the presence of high amplitude waves does not necessarily imply a nondiffusive character of wave-particle interactions. Previous estimates usually give threshold values for the wave amplitude such that trapping becomes possible\(^\text{1,9,42}\) but as we have shown, even for high amplitude waves, only over certain energy and pitch-angle ranges does nonlinear acceleration by trapping really become more effective than scattering. In case of whistler-mode waves with amplitudes \( \sim 50–100 \text{ mV/m} \) and propagation in the quasi-electrostatic mode for energies \( \in [30, 100] \text{ keV} \), the \( z_0 \)-range of prevalence of nonlinear acceleration is rather narrow \( \sim 5^\circ–10^\circ \). This range is much wider for small energy electrons \( \sim 10 \text{ keV} \).

It is interesting to note that the amplitude of regular energy jumps \( \langle \Delta\gamma \rangle \sim \gamma_0 \) and the probability of trapping \( \Pi \) depend similarly on the small parameter \( 1/k_0 \). Thus, the ratio of energy gained by particles due to trapping \( \Delta E_{\text{gain}} \sim \Pi \langle \Delta\gamma \rangle \) and lost by particles due to scattering \( \Delta E_{\text{lost}} \sim (1 - \Pi)\langle \Delta\gamma \rangle \) are of the same order: for \( 1/k_0 \ll 1 \) the ratio \( \Delta E_{\text{lost}}/\Delta E_{\text{gain}} \) depends on \( k_0 \) only through the combination \( \gamma k_0 \sim 1 \). Such a relationship between lost and gained energies was shown before in Ref. 44.

In this paper, we show that for high-amplitude oblique waves, the process of particle scattering is substantially modified in comparison with particle interaction with low-amplitude oblique waves in the quasi-linear regime. Even if the probability of trapping is small (or even zero, e.g., for \( \partial S/ \partial \tau < 0 \) the evolution of the particle energy due to scattering does not keep a diffusive character. There is a nonzero average value of the energy jumps \( \langle \Delta\gamma \rangle \neq 0 \). For long term dynamics, such jumps can be more important than diffusion due to \( \text{Var}(\Delta\gamma) \). They are responsible for a particle drift in energy space\(^\text{17} \) with a velocity \( \dot{\gamma} \approx \langle \Delta\gamma \rangle/T_{\text{bounce}} \), where \( T_{\text{bounce}} \) is the bounce period. For \( 100 \text{ keV} \) particles with \( z_0 \sim 45^\circ \) and \( 50 \text{ mV/m} \) wave amplitude, we can estimate this velocity as \( \dot{\gamma} \approx -2 \text{keV/s} \). This effect can be important for electron deceleration and related wave amplification (see discussion in

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**FIG. 8. Probability of trapping and ratio \( \Delta E_{\text{trap}}/\Delta E_{\text{scat}} > 1 \) for two energies and two wave amplitudes.**

We can also infer from the above estimates that the increase of \( L \)-shell value (i.e., the increase of \( R_0 \) and \( \omega_{pe} \) parameters) results in an increase of the critical wave amplitude. Thus, at larger \( L \), a higher wave intensity is necessary for nonlinear acceleration to prevail. This is an interesting and important result, because previous studies have shown that the increase of the \( L \)-shell value should correspond to an increase of the efficiency of nonlinear acceleration, i.e., an increase of \( \Delta\gamma_{\text{trap}} \) (see Ref. 6). Fig. 9 clearly demonstrates that this existing increase is nonetheless weaker than the increase of the efficiency of scattering in energy. However, it is worth noting too that the increase of \( L \) also results in a widening of the \( z_0 \)-range where resonant interaction is possible.\(^\text{6} \) As a result, the \( z_0 \)-range such that \( E_{\text{in}} < 100 \text{ mV/m} \) is actually increased at larger \( L \)-shells.
Ref. 31). The problem of nonlinear wave generation and amplification is very important for the planetary radiation belts, because purely linear instabilities seem to be unable to generate the observed high-amplitude waves.\textsuperscript{47,52}

For $\sim 100$ keV electrons at $L \sim 5$, this effect has the same magnitude for $50 \text{ mV/m}$ oblique whistler-mode waves as deceleration by phase bunching due to cyclotron-resonant interaction with high-amplitude ($\sim 60$ pT) parallel whistler-mode waves,\textsuperscript{2} but it should become stronger at lower energies, where it affects electrons with smaller equatorial pitch-angles $\alpha_0 < 45^\circ$. Since it acts against electron precipitation in the losscone (at very low $\alpha_0$), its coexistence at low electron energy with trapping acceleration could allow multiple (successive) accelerations before actual precipitation eventually occurs.\textsuperscript{5}

In conclusion, we have investigated the nonlinear scattering in energy of electrons resonantly interacting with high-amplitude oblique whistler-mode waves. A comparison of the efficiency of energy scattering and nonlinear acceleration by trapping shows that for reasonable wave amplitudes and electron energies $< 100$ keV, there is always some range of equatorial pitch-angles where nonlinear trapping prevails. This range is rather wide for small energy electrons $\sim 10$ keV but it shrinks as energy increases. We have derived analytical equations for the energy jumps due to nonlinear scattering. These expressions are valid for any system with inhomogeneous magnetic field and quasi-electrostatic waves.

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\begin{table}
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$L$ & $\omega_m$ & $E^*$ & $\alpha_0$ \\
\hline
4.5 & 0.35 & 10 keV & 150 mV/m \\
7.0 & 0.35 & 100 keV & 150 mV/m \\
4.5 & 0.2 & 100 keV & 150 mV/m \\
\hline
\end{tabular}
\caption{The critical wave amplitude is shown as a function of pitch-angle for several energies and three sets of system parameters.}
\end{table}


