Effective electrical conductivity of 3-D heterogeneous porous media

Sophie Hautot, Pascal Tarits

Effective electrical conductivity of 3-D heterogeneous porous media

Sophie Hautot and Pascal Tarits
UMR CNRS "Domaines Océaniques", IUEM-UBO, Brest, France

Received 13 February 2002; revised 6 May 2002; accepted 9 May 2002; published 18 July 2002.

[1] The relationship between the physical properties and the effective electrical conductivity of porous structures is studied using three-dimensional (3-D) models of random porosity. Synthetic electric (DC) and electromagnetic (EM) field data are calculated for a 3-D electrical conductivity model with a random porosity p ($p = 2 - 70\%$) embedded in a homogeneous half-space. The effective conductivity of the random porosity model is obtained from inversion of the synthetic data and agrees with a modified Archie's law. We applied percolation theory to our random porosity model to explain the variation of effective conductivity with p. We found that EM and DC data do not provide the same effective conductivity at a particular porosity but they do provide the same volume fraction of interconnected conductive elements. This volume fraction depends on the percolation threshold p_c. It follows a law of the form $\sim p$ above p_c and $\sim p^{2.2}$ below p_c. *INDEX TERMS:* 5109 Physical Properties of Rock: Magnetic and electrical properties; 5114 Physical Properties of Rock: Permeability and porosity; 3914 Mineral Physics: Electrical properties; 0925 Exploration Geophysics: Magnetic and electrical methods; 0644 Electromagnetics: Numerical methods

1. Introduction

[2] Determination of fundamental properties such as porosity, connectivity or permeability of underground porous material from geophysical investigations at the earth's surface relies upon laboratory measurements of small samples (0.01–0.1 m), and analogic and theoretical studies of rock properties. However, scaling the results from these studies to macroscopic bulk parameters obtained from geophysical data interpretation is problematic. There have been numerous attempts to reconcile laboratory measurements, field observations, and theoretical representation of heterogeneous structures using either effective medium theory or percolation theory [e.g., *David et al.*, 1990; *Haslund and Nøst*, 1998; *Mainprice*, 2000].

[3] Electrical conductivity is one of the most studied physical property of heterogeneous porous media. Relationships relating effective electrical conductivity σ_{eff} to physical properties of isotropic homogeneous rocks have been obtained for simple geometries [e.g., *Archie*, 1942; *Hashin and Shtrikman*, 1962; *Waff*, 1974]. Most theoretical studies of porous media have focussed on random networks properties in the case of extreme heterogeneities [e.g., *Shankland and Waff*, 1974; *Madden*, 1976; *Balberg et al.*, 1991]. While laboratory measurements of σ_{eff} of porous rock samples are in general adequately modeled with Archie's law [e.g., *Brace et al.*, 1965], statistical rock models fail to reproduce this relationship unless interconnection is assumed at all scales, from $\leq$ μm to $\geq$ cm [*Madden*, 1976; *Wong et al.*, 1984].

[4] The modeling of electric (DC) or electromagnetic (EM) soundings data provides subsurface conductivity structures that can be interpreted in terms of σ_{eff} of the medium. Physical properties are then obtained from relationships derived from Archie's law or Hashin and Shtrikman's model [e.g., *Shankland and Waff*, 1974; *Flóvenz et al.*, 1985]. An improved model should include realistic porous structures in the three-dimensional (3-D) DC or EM algorithms used to model the field data in order to better describe the transport properties. *Bigalke* [1999, 2000] has recently used a 3-D forward DC solver [*Spitzer*, 1995] to model electric properties of heterogeneous media at the sample scale.

[5] In this study, we used percolation theory and 3-D DC and EM modeling to demonstrate that σ_{eff} of simple porous models is controlled by an effective interconnected volume fraction (EIVF) that depends on the percolation threshold p_c. First, we generated random porosity models at the meter scale. The models were converted into an electrical conductivity structure embedded into a conductive half-space. We then obtained σ_{eff} from DC and EM data modeled at the surface of the half-space. We found that while σ_{eff} obtained from EM and DC data are different at a given porosity, they are controlled by the same EIVF.

2. Conductivity Model and Properties of the Porous Medium

[6] The porosity model is a 2 × 2 × 2 meter cube comprised of 8000 cubic 0.1 m side cells. Each cell is either electrically resistive with conductivity σ_s ($\sigma_s = 6.25 \times 10^{-4}$ S/m) or conductive with conductivity σ_f ($\sigma_f = 0.2$ S/m) (Figure 1). The pores are the conductive cells. The porosity p is the number of conductive cells over the total number of cells. The number of conductive cells selected randomly is increased until a given porosity is reached. Five porosity models are randomly generated for each p. p ranges between 2 and 70%. The cube is embedded in a homogeneous half-space with conductivity σ_{hs} ($\sigma_{hs} = 0.01$ S/m).

[7] Percolation theory is applied to the random porosity model to describe the network structure. Percolation theory indicates that at p_c the properties of the network change abruptly [e.g., *Stauffer and Aharony*, 1992]. As the volume fraction p of conductive elements increases, they form clusters of interconnected elements. The value p_c is the volume fraction of elements at which the largest cluster

0094-8276/02/2002GL014907

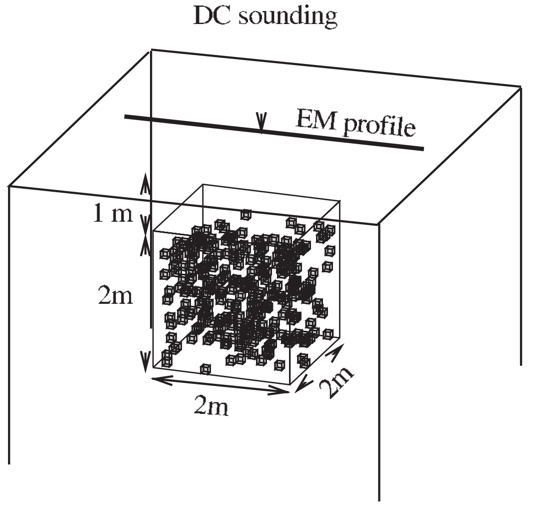

Figure 1. Heterogeneous porous medium embedded in a homogeneous half-space (hs) host-medium (σ_{hs} = 0.01 *S/m*).

spans the medium. In principle percolation theory applies to infinite systems. For finite-size systems, the results may be applied although p is no longer a step function at p_c [*Stauffer and Aharony*, 1992]. For our random porosity model p_c = 0.33, in good agreement with the theoretical value (0.31) for a 3-D cubic array in the site percolation case [*Stauffer and Aharony*, 1992].

[8] The size of individual clusters (excluding the percolating cluster above p_c) is given by the radius of gyration R_s as $R_s^2 = \Sigma|r_i - r_0|^2/s$ [*Stauffer and Aharony*, 1992], where r_i is the position of each element of the cluster, r_0 is the center of mass of the cluster ($r_0 = \Sigma\ r_i/s$), and s is the number of elements in the cluster. All dimensions are normalized and are in the range 0–1. The connectivity (or correlation) length ξ is the average distance between any two conductive elements in a cluster, averaged over all clusters except the percolating one. In percolation theory the value ξ corresponds to an average cluster diameter [*Stauffer and Aharony*, 1992]. For 3-D networks ξ follows a power law $\propto |p - p_c|^{-0.875}$ [*Berkowitz and Ewing*, 1998]. Figure 2 presents ξ for the random model at 10 porosity values, compared to the theoretical value. Our results fit the law predicted by percolation theory reasonably well. We use ξ as a connectivity threshold at a given p. Clusters with $2R_s \geq \xi$ form fully interconnected networks while clusters with $2R_s < \xi$ are isolated pockets.

3. Effective Conductivity

[9] The simulation of EM and DC surface data was carried out with 3-D DC and EM solvers [*Spitzer*, 1995; *Mackie et al.*, 1993] at the surface of the host-medium (Figure 1). The DC results were obtained for a Schlumberger electrode array configuration centered above the buried cube. The EM results were obtained for a plane wave inducing field at the VLF (very low frequency) radio transmitter frequency 20 kHz along a profile above the cube (Figure 1). To get the value σ_{eff} we set the cube homogeneous with a constant conductivity. The conductivity for which the DC or EM response of this new model fits the data obtained for the heterogeneous cube is σ_{eff}. This conductivity was obtained by inverting the surface data modeled from the heterogeneous cube. The rest of the model parameters (the size and depth of the cube and the half-space conductivity) are unchanged. The inversion scheme is described in *Hautot et al.* [2000; 2002].

[10] The scheme used to obtain σ_{eff} is validated for exact geometrical models [*Waff*, 1974]. These models have regularly spaced resistive and conductive cells to form homogeneous fully-connected (HFC) or fully-disconnected (HFD) networks. The HFD model is comprised of regularly distributed isolated conductive cells while the HFC model is comprised of regularly distributed isolated resistive cells. We generated seven HFD and HFC models with different isolated cells densities. For these models p varies from 2.7–97%. Figure 3 compares the σ_{eff} obtained with the theoretical values. The agreement is good for HFC models (from EM and DC results) and the HFD model (from EM results). The values σ_{eff} for the HFD model (from DC results) is slightly more conductive than the theoretical model and corresponds to Waff's HFD formula when $\sigma_s = 8.3 \times 10^{-4}$ S/m instead of the real value (6.25×10^{-4} S/m).

[11] The random porosity model σ_{eff} (σ_{eff}^{dc} from DC results and σ_{eff}^{em} from EM results) is also presented in Figure 3. The values σ_{eff}^{dc} and σ_{eff}^{em} are different and vary between HFC and HFD values. In the random porosity model with p = 70% there is one single cluster spanning all the edges of the cube and both σ_{eff}^{dc} and σ_{eff}^{em} coincide with the HFC values. At p = 2% the clusters are isolated pockets and have a small volume (maximum of 3 conductive elements per cluster) and, both σ_{eff}^{dc} and σ_{eff}^{em} coincide with the HFD values.

[12] Theoretically, for p between 2% and 70%, σ_{eff}^{dc} and σ_{eff}^{em} should be identical because similar effects control the distortion of the EM and DC fields by the heterogeneous cube. The distortion of the DC field is the result of electric charges on the conductivity contrasts within, and at the surface of the cube [*Telford et al.*, 1990]. At the scale of the conductivity model and, for an EM inducing field at 20 kHz, self induction is negligible [*Le Mouël and Menvielle*, 1982], hence the perturbation of the induced electric field

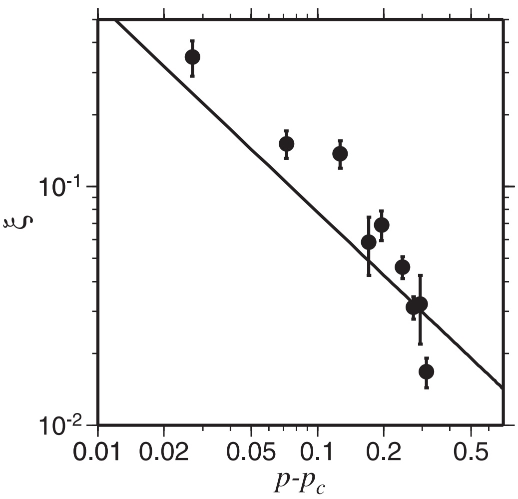

Figure 2. Connectivity length ξ vs. $p-p_c$. Mean values and error bars are calculated from five porosity models. Solid line: power law $|p-p_c|^{-0.875}$.

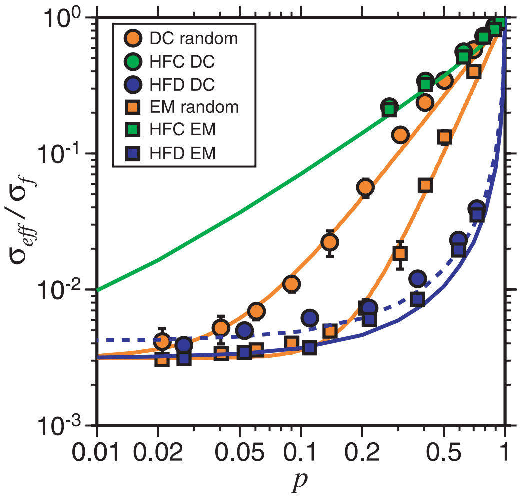

Figure 3. σ_{eff}/σ_f vs. porosity p. Symbols are for numerical σ_{eff} and lines for analytical expressions. Mean values and error bars are calculated from five porosity models. Green line: *Waff*'s [1974] HFC bound. Blues lines: HFD bound (solid: $\sigma_s = 6.25 \times 10^{-4}$ S/m, dashed: $\sigma_s = 8.4 \times 10^{-4}$ S/m). Orange lines: modified Archie's law.

by the cube is also controlled by the quasi-static charges on the conductivity contrasts.

[13] Both DC and EM algorithms are based on finite-difference equations. They differ in the gridding of the conductivity model and in the calculation of the field at each grid point. In the DC algorithm [*Spitzer*, 1995], an equivalent parallel circuit is used at each grid point to approximate the conductivity contrasts (by averaging the conductivity and its gradients over the 8 neighbor cells). In the EM algorithm [*Mackie et al.*, 1993], an equivalent serial circuit is used with averaging the resistivity (the reciprocal of conductivity) between two adjacent cells to insure the continuity of the electric current normal to the cells' faces. In both cases, contrasts in the conductivity model are smoothed by the averaging process at the scale of the grid point distance. When this distance is much smaller than the size of the smallest heterogeneity in the conductivity model, the approximations are equivalent and the conductivity contrast is accurately modeled. In our conductivity models the minimum grid-point distance was set to 0.05 m for computational reasons. The distance is not small compared to the smallest heterogeneity (0.1 m), so the averaging affects the results. The conductivity model is smoother than the input model (with sharp conductivity contrasts) and is more resistive in EM calculations than in DC. The models run with this grid result in σ_{eff}^{dc} slightly larger than σ_{eff}^{em} for the HFD models and in different σ_{eff}^{dc} and σ_{eff}^{em} for the random porosity model (Figure 3). The latter is more resistive than the former because of the difference in the numerical approximation (parallel and serial).

4. Effective Conductivity and Percolation Threshold

[14] Neither the σ_{eff}^{dc} nor the σ_{eff}^{em} curve from the random porosity model presents a percolation threshold at $p_c = 0.33$ (Figure 3). Both σ_{eff} approximatively follow Archie's law for $p > 0.1$–0.2. Archie's law is on the form $\sigma_{eff} = \sigma_f p^m$ (m is an empirical factor with a value $\sim$1.3–4 [e.g., *Sen et al.*, 1981]). Both σ_{eff} curve at lower p due to $\sigma_s \neq 0$. *Hermance* [1979] has proposed a modified Archie's law to account for $\sigma_s \neq 0$ of the form $\sigma_{eff} = (\sigma_f - \sigma_s)\, p^m + \sigma_s$. Both σ_{eff} fit well this model with exponents $m = 1.9$ for DC models and $m = 3.3$ for EM models, in the range of observed m values in laboratory studies [*Sen et al.*, 1981]. Models with $\sigma_s = 6.25 \times 10^{-5}$ $-$ 6.25×10^{-6} (for which the modified Archie's law is very close to Archie's law) were run and σ_{eff} fitted Archie's law with the same exponents m as before.

[15] Discrepancies between percolation theory applied to conductivity models and experimental data have been previously reported [e.g., *Madden*, 1976; *Wong et al.*, 1984; *Pham*, 2000]. On one hand, the theory for random electric networks predicts that, slightly above p_c, σ_{eff} varies according to $| p - p_c|^{\mu}$ (μ is a critical exponent $\sim$2 in 3-D [*Stauffer and Aharony*, 1992]), in disagreement with our results since σ_{eff} follows the modified Archie's law. On the other hand, in percolation theory Archie's law (for the limiting case $\sigma_s = 0$) implies $p_c = 0$ which is not a characteristic of our models. As suggested by *Shankland and Waff* [1974] and *Madden* [1976], Archie's law here seems to be a description of the electric signature of the statistical properties of the porous medium rather than an intrinsic characteristic of the models.

5. Effective Interconnected Volume

[16] The examination of the structure of the clusters in the random porosity model shows that below p_c, there are fully connected (FC) and fully disconnected (FD) clusters at all porosities except the smallest, while above p_c the structure is dominated by the percolating FC cluster. At a given p, an effective interconnected volume fraction (EIVF) p_ξ is defined as follows. At $p < p_c$, p_ξ is the sum over the volumes of the clusters of conductive elements with a diameter of gyration $2R_s \geq \xi$. At $p > p_c$, p_ξ is the difference between p and the sum of all isolated pockets that forms a

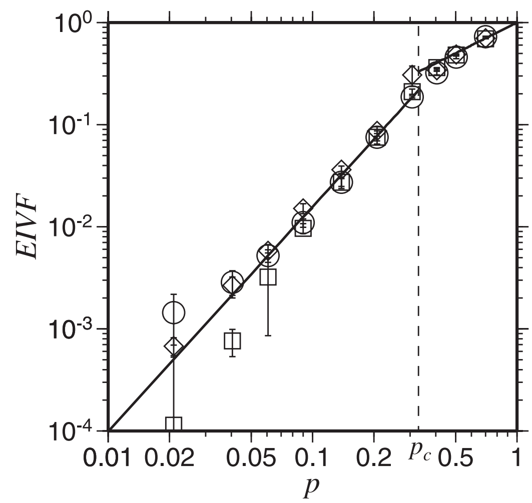

Figure 4. Effective interconnected volume fraction (*EIVF*) vs. p. Symbols: p_ξ diamond; p_a circle; $p_{a'}$ square. Above p_c, the slope of the thin line is 1, beneath p_c, the slope is 2.2.

fully disconnected volume fraction p_{FD}, $p_\xi = p - p_{FD}$. The values of p_ξ versus p are presented in Figure 4. Above p_c, $p_\xi \simeq p$ and indicates that almost all the conductors are fully interconnected while at $p < p_c$, $p_\xi \simeq p^{2.2}$ which accounts for the balance between FC and FD volume fractions.

[17] The FC and FD volume fractions in a random porosity model at p are electrically connected through a circuit characterized by σ_{eff} equal to either σ_{eff}^{dc} or σ_{eff}^{em}, depending on the nature of the approximation used in the numerical solutions (serial for EM and parallel for DC). The two end-members of σ_{eff} are HFC (σ^+) and HFD (σ^-) values. The relationship between either σ_{eff}^{dc} or σ_{eff}^{em} and σ^+ and σ^- is different for serial or parallel circuits:

$$\sigma_{dc}^{eff} = f_a \sigma^+ + (1 - f_a)\sigma^-, \tag{1}$$

$$\sigma_{em}^{eff} = (f_{a'}/\sigma^+ (1 - f_{a'})/\sigma^-)^{-1}. \tag{2}$$

The terms f_a and $f_{a'}$ are the volume fractions of interconnected conducting elements at p for DC and EM, respectively. The volume fractions p_a and $p_{a'}$ are obtained from $p_a = p \times f_a$ and $p_{a'} = p \times f_{a'}$ and should be equal to p_ξ. These volume fractions are calculated for each σ_{eff} at all p and are also presented in Figure 4. For $p > 5\%$, p_a and $p_{a'}$ are nearly identical and are a very good fit to p_ξ above as well as below p_c except at the smallest p values. This result demonstrates that despite the fact that p_c was not evident in σ_{eff} (Figure 3), it controls the variation of σ_{eff} with p for our random porosity model.

6. Conclusion

[18] Percolation theory seems a useful tool to explain some characteristics of the distortion of observed EM fields [e.g., *Bahr*, 2000]. However, it is difficult to relate directly transport properties in highly heterogeneous media to field observations because 3-D DC and EM algorithms cannot describe the finest scale structures. While our random conductivity models are the simplest because of this limitation, they reproduce reasonably well the scale independent properties predicted by percolation theory. As a result, it was possible to interpret synthetic EM and DC data in terms of percolation models. We defined a new parameter, the EIVF which is closely related to transport properties. We found that p_c may be derived from the EIVF. In practice, the EIVF could be obtained from observed σ_{eff} as a function of porosity, either at laboratory or field scales. The EIVF would provide a better description of the transport properties than the classical effective medium parameters such as Archie's law. For real rocks the heterogeneous conductivity models should account for fracture porosity and surface conductivity.

[19] **Acknowledgments.** Sophie Hautot was at the Department of Geology and Geophysics, University of Edinburgh (UK), while part of this work was undertaken. S.H. was supported by the Region Bretagne and Pays de la Loire. We are grateful to Cara Wilson for careful reading of the manuscript, and to Karsten Bahr and one anonymous referee for constructive criticism.

References

Archie, G. E., The electrical resistivity log as an aid in determining some reservoir characteristics, *Trans. Am. Inst. Mech. Eng.*, *146*, 54–61, 1942.
Bahr, K., Percolation in the crust derived from distortion of electric fields, *Geophys. Res. Lett.*, *27*, 1049–1052, 2000.
Balberg, I., B. Berkowitz, and G. E. Drachsler, Application of a percolation model to flow in fractured hard rocks, *J. Geophys. Res.*, *96*, 10,015–10,021, 1991.
Berkowitz, B., and R. P. Ewing, Percolation theory and network modeling — Applications in soil physics, *Surv. Geophys.*, *19*, 23–72, 1998.
Bigalke, J., Investigation of the conductivity of random networks, *Physica A*, *272*, 281–293, 1999.
Bigalke, J., A study concerning the conductivity of porous rocks, *Phys. Chem. Earth A*, *25*, 189–194, 2000.
Brace, W. F., A. S. Orange, and T. R. Madden, The effect of pressure on the electrical resistivity of water-saturated crystalline rocks, *J. Geophys. Res.*, *70*, 5669–5678, 1965.
David, C., Y. Gueguen, and G. Pampoukis, Effective medium theory and network theory applied to the transport properties of rock, *J. Geophys. Res.*, *95*, 6993–7005, 1990.
Flóvenz, Ó., L. S. Georgsson, and K. Árnason, Resistivity structure of the upper crust in Iceland, *J. Geophys. Res.*, *90*, 10,136–10,150, 1985.
Hashin, Z., and S. Shtrikman, A variational approach to the theory of the effective magnetic permeability of multiphase materials, *J. Appl. Phys.*, *33*, 3125–3131, 1962.
Haslund, E., and B. Nøst, Determination of porosity and formation factor of water-saturated porous specimens from dielectric dispersion measurements, *Geophysics*, *63*, 149–153, 1998.
Hautot, S., P. Tarits, K. Whaler, B. Le Gall, J. J. Tiercelin, and C. Le Turdu, The deep structure of the Baringo Rift basin (central Kenya) from 3-D magnetotelluric imaging: Implications for rift evolution, *J. Geophys. Res.*, *105*, 23,493–23,518, 2000.
Hautot, S., P. Tarits, F. Perrier, C. Tarits, and M. Trique, Groundwater electromagnetic imaging in complex geological and topographical regions: A case study of a tectonic boundary in the French Alps, *Geophysics*, in press, 2002.
Hermance, J. F., The electrical conductivity of materials containing partial melt, a simple model from Archie's law, *Geophys. Res. Lett.*, *6*, 613–616, 1979.
Le Mouël, J. L., and M. Menvielle, Geomagnetic variation anomalies and deflection of telluric currents, *Geophys. J. R. Astron. Soc.*, *68*, 575–587, 1982.
Mackie, R. L., T. R. Madden, and P. E. Wannamaker, Three-dimensional magnetotelluric modeling using difference equations — Theory and comparisons to integral equation solutions, *Geophysics*, *58*, 215–226, 1993.
Madden, T. R., Random networks and mixing laws, *Geophysics*, *41*, 1104–1125, 1976.
Mainprice, D., The estimation of seismic properties of rocks with heterogeneous microstructures using a local cluster model — Preliminary results, *Phys. Chem. Earth A*, *25*, 155–161, 2000.
Pham, D. C., Electrical properties of sedimentary rocks having interconnected water-saturated pore spaces, *Geophysics*, *65*, 1093–1097, 2000.
Sen, P. N., C. Scala, and M. H. Cohen, A self-similar model for sedimentary rocks with application to the dielectric constant of fused glass beads, *Geophysics*, *46*, 781–795, 1981.
Shankland, T. J., and H. S. Waff, Conductivity in fluid-bearing rocks, *J. Geophys. Res.*, *79*, 4863–4868, 1974.
Spitzer, K., A 3-D finite-difference algorithm for DC resistivity modelling using conjugate gradient methods, *Geophys. J. Int*, *123*, 903–914, 1995.
Stauffer, D., and A. Aharony, *Introduction to Percolation Theory*, 2nd ed., 181 pp., Taylor and Francis, London, 1992.
Telford, W. M., L. P. Geldart, and R. E. Sheriff, *Applied Geophysics*, 2nd ed., 770 p., Cambridge University Press, 1990.
Waff, H. S., Theoretical considerations of electrical conductivity in a partially molten mantle and implications for geothermometry, *J. Geophys. Res.*, *79*, 4003–4010, 1974.
Wong, P., J. Koplik, and J. P. Tomanic, Conductivity and permeability of rocks, *Phys. Rev. B Solid State*, *50*, 6606–6614, 1984.

S. Hautot and P. Tarits, IUEM-UBO, UMR CNRS 6538, Place Nicolas Copernic F-29280, Plouzané, France. (hautot@sdt.univ-brest.fr; tarits@sdt.univ-brest.fr)