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Institut d'Astrophysique et Géophysique, Université de Liège, 5 *Avenue de Cointe,* B-4000 *Liège, Belgium. E-mail: rogister@astro.ulg.ac.be*

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SUMMARY

The poor results of Smith's (1974, 1977) computations concerning the motions of the solid inner core of a slowly rotating, ellipsoidal earth model can be explained by the fact that Smith neglected in the equations of motion second-order terms in the ellipticity. We demonstrate this statement by calculating the eigenfunctions associated with the diurnal tilt-over mode (TOM) and inserting them into the equations of motion. Next, we compute the free core nutation (FCN) and the free inner core nutation (FICN) for various realistic earth models. We find that their respective periods are in good agreement with the results of Mathews *et al.* (1991b) and de Vries & Wahr (1991), who have computed these modes by other means.

Key words: Earth's rotation, free oscillations, nutation.

1 INTRODUCTION

Analytical and numerical computations of the free modes of a rotating, deformable earth model of ellipsoidal shape are far more complicated than those of a non-rotating model of spherical shape. The influence of rotation and ellipticity on the usual seismic normal modes with periods smaller than 1 hr can be studied by perturbation methods (Backus & Gilbert 1961; Dahlen 1968, 1969; Dahlen & Sailor 1979). In this case, there is no need to solve the equations of motion for an ellipsoidal model. However, if we wish to deal with the free wobbles or free nutations of the Earth, we can no longer treat rotation and ellipticity as perturbations. Indeed, those motions are *not* free motions of a spherical model.

The first studies of these motions for a deformable earth model were performed by Bryan (1889) and Hough (1895, 1896) towards the end of the 19th century. Owing to the simplicity of the model considered, namely a homogeneous, incompressible, liquid ellipsoid, Bryan (1889) was able to solve exactly the equations of motion by using spheroidal harmonics and elliptic coordinates. Hough (1896), considering a homogeneous, incompressible, solid ellipsoid, made the *a priori* assumption that the displacement field was the sum of a rigid rotation and a smaller elastic displacement. He determined the latter by neglecting the inertia forces and ellipticity in the equations of motion. Expressing then the conservation of the angular momentum for the whole body, he obtained an algebraic equation for the free period. Roughly speaking, we may say that these two methods are at the root of the main approaches for computing the free wobbles of rotating bodies.

The former method has, in principle, the advantage of allowing for the computation of *all* the normal modes of the reference body, but it has to face difficult problems when the body, assumed to be initially in hydrostatic equilibrium, is heterogeneous and, as a consequence, the ellipticity of its strata varies along the radius. Indeed, in this case there exists neither a suitable coordinate system nor appropriate base functions. Therefore, the equations of motion are generally solved in spherical coordinates, using spherical harmonic functions. Displacements associated with different harmonic degrees are then coupled to each other and, in order to obtain numerical or analytic solutions, the chain coupling must be truncated at a given level. This truncation is sometimes questionable.

On the other hand, the latter method is designed to study nutational motions only. It yields analytical expressions for the proper periods, but relies on the assumption that the elastic deformation is smaller than the rotational displacement.

In this paper, I shall focus on the theory of Smith (1974), who derived the scalar equations governing the motion of a slowly rotating, slightly ellipsoidal, elastic body. Later, Smith (1977) published the results obtained by numerically solving these equations. While the theory worked rather well for determining the periods of the Chandler wobble (CW) and the nearly diurnal free wobble (NDFW), also known as the free core nutation (FCN), the motions of the solid inner core were not described properly. In particular, the period of the inner core equivalent of the Chandler wobble (named 'Chandler wobble of the inner core' by Smith) did not agree at all with the period computed by Busse (1970) for a very simple earth model. Moreover, Smith discovered a free nutation of the inner core (FICN) but, in view of his previous inconclusive results, he refused to give much credit to his finding. He believed that the reason for the failure of the theory in the inner core was the insufficient number of terms retained in the coupling chain.

Later, Wahr (1981a,b,c) exploited Smith's theory intensively to compute the forced motions of an elliptical earth. Extending the theory, Dehant (1987) allowed for material anelasticity in the equations of motion, but used essentially Wahr's numeric code based on the equations of Smith (1974). Recently, Schastok (1997) argued that second-order terms in the ellipticity must be kept in the equations of motion and the boundary conditions, even if the ellipticity is as small as that of the Earth. Smith and his followers, especially Wahr & Dehant, had neglected such second-order terms. Schastok's argument was founded on the analytical form of the solution of the deformation equations of a simple earth model. Thus, taking also into account the non-hydrostatic structure of the Earth, he provided a new nutation series. In particular, he found a period for the free core nutation in agreement with the observations, and he mentioned a period for the free inner core nutation close to that calculated by de Vries & Wahr (1991) and by Mathews *et al.* (1991b). The latter papers extend the theory of Sasao *et al.* (1980), which itself belongs to the second type of approach briefly described above.

In the following, I shall give a different proof that second-order terms in the ellipticity must be retained in the system of differential equations of Smith (1974) in order to provide significant results. My proof consists of showing that the deformation equations yield the correct eigenperiod and eigenfunctions of a specific mode of any elliptical earth model, namely the tilt-over mode (TOM), only if second-order terms are accounted for. The TOM is merely a rigid rotation of the body about an axis that does not coincide with the space-fixed z-axis of the uniformly rotating reference frame. In Section 2, I first establish a formula yielding the variation of the gravity potential induced by the rigid nutation of an ellipsoidal homogeneous model. In Section 3, I then summarize the important steps of Smith's (1974) theory, and I introduce the known TOM eigenperiod and eigenfunctions into the equations of motion to demonstrate that they are not correct solutions of the problem unless second-order terms in the ellipticity are retained. This result invalidates part of the scheme of numerical integration of the deformation equations written by Smith. Finally, in Section 4, I compare my numerical results for the FCN and the FICN with those of de Vries & Wahr (1991) and Mathews *et al.* (1991b), and I provide plots of the eigenfunctions of both of these modes.

2 RIGID NUTATION OF A SLIGHTLY SPHEROIDAL UNIFORM BODY

2.1 Gravity potential of a slightly spheroidal uniform body

Let us consider a slightly spheroidal homogeneous body, possessing a density ρ_0. The term 'spheroid' designates here an ellipsoid that has its symmetry of revolution about the polar axis. In the following, we shall use the words 'ellipsoid' and 'spheroid' interchangeably, it being understood that the body has the shape of a spheroid. 'Slightly' means that the difference in lengths between the major and minor axes is small compared to the lengths of any of these axes. We define a right-handed reference system such that the origin O is at the centre of mass of the body, the Oz-axis is directed along the minor axis, the axis of symmetry of the body, and the Ox- and Oy-axes lie in the equatorial plane. In this reference system, spherical coordinates are denoted by r, θ and φ. To first order in the flattening, the gravity potential inside (ϕ_i) and outside (ϕ_e) the body may be written as

$$\phi_\mathrm{i} = 4\pi G\rho_0\left[\frac{r^2}{6} - \frac{R^2}{2} - \frac{r^2}{5}\, s_2 P_2(\cos\theta)\right] \tag{1}$$

and

$$\phi_\mathrm{e} = 4\pi G\rho_0\left[-\frac{R^3}{3r} - \frac{R^5}{5r^3}\, s_2 P_2(\cos\theta)\right], \tag{2}$$

where R is the radius of the equivolumetric sphere, P_2 is the Legendre polynomial of degree 2 and s_2 is a figure function related to the flattening ε by

$$s_2 = -\frac{2}{3}\,\varepsilon\,. \tag{3}$$

If the body was not homogeneous, eq. (3) would still hold but s_2 and ε would be functions of r. The continuity conditions to be satisfied by the potential may be applied either at the surface of the ellipsoid,

$$r = R(1 + s_2 P_2)\,, \tag{4}$$

or at the surface of the equivolumetric sphere of radius R, i.e. $r = R$. In the first case, we have

$$\begin{aligned}\phi_\mathrm{i}(R + Rs_2P_2) &= \phi_\mathrm{e}(R + Rs_2P_2)\\ &= 4\pi G\rho_0 R^2\left(-\frac{1}{3} - \frac{2}{15}\, s_2 P_2\right)\end{aligned} \tag{5}$$

and

$$\begin{aligned}&-\nabla\phi_\mathrm{i}(R + Rs_2P_2)\\ &\quad= -\nabla\phi_\mathrm{e}(R + Rs_2P_2)\\ &\quad= 4\pi G\rho_0 R\left[\left(-\frac{1}{3} + \frac{1}{15}\, s_2 P_2\right)\mathbf{e}_r + \frac{3}{5}\, s_2 \cos\theta\sin\theta\,\mathbf{e}_\theta\right].\end{aligned} \tag{6}$$

In the second case, at $r = R$, ϕ is still continuous,

$$\phi_\mathrm{i}(R) = \phi_\mathrm{e}(R) = 4\pi G\rho_0 R^2\left(-\frac{1}{3} - \frac{1}{5}\, s_2 P_2\right), \tag{7}$$

but the normal component of the gravity is discontinuous,

$$-\nabla\phi_\mathrm{i}(R) = 4\pi G\rho_0 R\left[\left(-\frac{1}{3} + \frac{2}{5}\, s_2 P_2\right)\mathbf{e}_r + \frac{3}{5}\, s_2\cos\theta\sin\theta\,\mathbf{e}_\theta\right] \tag{8}$$

and

$$-\nabla\phi_\mathrm{e}(R) = 4\pi G\rho_0 R\left[\left(-\frac{1}{3} - \frac{3}{5}\, s_2 P_2\right)\mathbf{e}_r + \frac{3}{5}\, s_2\cos\theta\sin\theta\,\mathbf{e}_\theta\right]. \tag{9}$$

Therefore, we recover the usual boundary condition to be imposed on the gravity of a deformed spherical body, namely

$$(\nabla\phi_i + 4\pi G\rho_0 \mathbf{u})\cdot\mathbf{e}_r = \nabla\phi_e\cdot\mathbf{e}_r\,, \tag{10}$$

provided the displacement is

$$\mathbf{u} = Rs_2P_2\mathbf{e}_r\,. \tag{11}$$

2.2 Geometrical description of a rigid nutation

Let us now suppose that the ellipsoidal body undergoes a small rigid rotation through an angle β about an axis that is in the equatorial plane of the undisturbed body. We define a second system of reference, $Ox'y'z'$, obtained by rotating the $Oxyz$ system with the body. In $Ox'y'z'$, the spherical coordinates are denoted by r, θ' and φ'. The Euler angles are then α, β and $-\alpha$, where α is the angle between Ox and the axis of rotation, which is thus also the line of nodes. They correspond to a rotation through the angle α about Oz, followed by a rotation through the angle β about Ox, and a rotation through the angle $-\alpha$ about Oz'. The basis vectors $\mathbf{e}_{x'}$, $\mathbf{e}_{y'}$, $\mathbf{e}_{z'}$ can be expressed as functions of $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ by means of the matrix operation

$$\begin{pmatrix}\mathbf{e}_{x'}\\ \mathbf{e}_{y'}\\ \mathbf{e}_{z'}\end{pmatrix} = \begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\beta & \sin\beta\\ 0 & -\sin\beta & \cos\beta\end{pmatrix} \times \begin{pmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{e}_x\\ \mathbf{e}_y\\ \mathbf{e}_z\end{pmatrix}. \tag{12}$$

As β is infinitesimal, this relation simplifies to

$$\begin{pmatrix}\mathbf{e}_{x'}\\ \mathbf{e}_{y'}\\ \mathbf{e}_{z'}\end{pmatrix} = \begin{pmatrix}1 & 0 & -\beta\sin\alpha\\ 0 & 1 & \beta\cos\alpha\\ \beta\sin\alpha & -\beta\cos\alpha & 1\end{pmatrix}\begin{pmatrix}\mathbf{e}_x\\ \mathbf{e}_y\\ \mathbf{e}_z\end{pmatrix}, \tag{13}$$

or, on inverting,

$$\begin{pmatrix}\mathbf{e}_x\\ \mathbf{e}_y\\ \mathbf{e}_z\end{pmatrix} = \begin{pmatrix}1 & 0 & \beta\sin\alpha\\ 0 & 1 & -\beta\cos\alpha\\ -\beta\sin\alpha & \beta\cos\alpha & 1\end{pmatrix}\begin{pmatrix}\mathbf{e}_{x'}\\ \mathbf{e}_{y'}\\ \mathbf{e}_{z'}\end{pmatrix}. \tag{14}$$

The sine and cosine of the spherical coordinates θ' and φ' can then be expressed as functions of θ and φ by substituting the relations (14) into

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z = x'\mathbf{e}_{x'} + y'\mathbf{e}_{y'} + z'\mathbf{e}_{z'} \tag{15}$$

and equating the terms proportional to $\mathbf{e}_{x'}$, $\mathbf{e}_{y'}$, $\mathbf{e}_{z'}$. In this way we find

$$\cos\theta' = \cos\theta + \beta\sin\theta\sin(\alpha-\varphi)\,, \tag{16}$$

$$\sin\theta' = \sin\theta - \beta\cos\theta\sin(\alpha-\varphi)\,, \tag{17}$$

$$\cos\varphi' = \cos\varphi\left[1 + \beta\,\frac{\cos\theta}{\sin\theta}\,\sin(\alpha-\varphi)\right] - \beta\,\frac{\cos\theta}{\sin\theta}\,\sin\alpha\,, \tag{18}$$

$$\sin\varphi' = \sin\varphi\left[1 + \beta\,\frac{\cos\theta}{\sin\theta}\,\sin(\alpha-\varphi)\right] + \beta\,\frac{\cos\theta}{\sin\theta}\,\cos\alpha\,. \tag{19}$$

The basis vector $\mathbf{e}_{\theta'}$ is given by

$$\begin{aligned}\mathbf{e}_{\theta'} &= \cos\theta'\cos\varphi'\mathbf{e}_{x'} + \cos\theta'\sin\varphi'\mathbf{e}_{y'} - \sin\theta'\mathbf{e}_{z'}\\ &= \cos\theta\cos\varphi\mathbf{e}_x + \cos\theta\sin\varphi\mathbf{e}_y - \sin\theta\mathbf{e}_z\\ &\quad + \frac{\beta}{\sin\theta}\{[-\sin\alpha + \cos\varphi\sin(\alpha-\varphi)]\mathbf{e}_x\\ &\quad + [\cos\alpha + \sin\varphi\sin(\alpha-\varphi)]\mathbf{e}_y\}\,.\end{aligned} \tag{20}$$

Thus, we obtain to first order in β

$$\mathbf{e}_{\theta'} = \mathbf{e}_\theta + \frac{\beta}{\sin\theta}\cos(\alpha-\varphi)\mathbf{e}_\varphi\,. \tag{21}$$

Since the rotation through the angle β is infinitesimal, it can be represented by a vector,

$$\boldsymbol{\beta} = \beta(\cos\alpha\mathbf{e}_x + \sin\alpha\mathbf{e}_y)\,, \tag{22}$$

and the displacement vector of a particle is

$$\mathbf{u} = \boldsymbol{\beta}\times\mathbf{r} = r\beta[\sin(\alpha-\varphi)\mathbf{e}_\theta - \cos\theta\cos(\alpha-\varphi)\mathbf{e}_\varphi]\,. \tag{23}$$

This is easily seen to be a toroidal displacement of degree 1 and order 1:

$$\mathbf{u} = \frac{1}{\sin\theta}\frac{\partial\overline{W}}{\partial\varphi}\,\mathbf{e}_\theta - \frac{\partial\overline{W}}{\partial\theta}\,\mathbf{e}_\varphi\,, \tag{24}$$

where

$$\overline{W} = r\beta\sin\theta\cos(\alpha-\varphi)\,. \tag{25}$$

2.3 Variation of the gravity potential of a slightly spheroidal homogeneous body undergoing a rigid nutation

The gravity potential ϕ' of the tilted body described in Section 2.2 is obviously of the form (1) or (2), with θ replaced by θ'. The relations (16) and (17) allow us to write

$$\phi'_i = 4\pi G\rho_0\left\{\frac{r^2}{6} - \frac{R^2}{2} - \frac{r^2}{5}\,s_2[P_2(\cos\theta) + 3\beta\cos\theta\sin\theta\sin(\alpha-\varphi)]\right\}, \tag{26}$$

$$\phi'_e = 4\pi G\rho_0\left\{-\frac{R^3}{3r} - \frac{R^5}{5r^3}\,s_2[P_2(\cos\theta) + 3\beta\cos\theta\sin\theta\sin(\alpha-\varphi)]\right\}. \tag{27}$$

Thus, the variations of the gravity potential $\phi_1 = \phi' - \phi$ caused by the rigid nutation inside and outside the body become

$$\phi_{1i} = 4\pi G\rho_0\left[-\frac{3r^2}{5}\,s_2\beta\cos\theta\sin\theta\sin(\alpha-\varphi)\right], \tag{28}$$

$$\phi_{1e} = 4\pi G\rho_0\left[-\frac{3R^5}{5r^3}\,s_2\beta\cos\theta\sin\theta\sin(\alpha-\varphi)\right]. \tag{29}$$

They are harmonic functions of degree 2 and order 1. The quantity ϕ_1 is continuous at $r = R$ or at the ellipsoidal surface (eq. 4). It is proportional to s_2 and β and, of course, vanishes for a spherical body. The gravity is

$$-\boldsymbol{\nabla}\phi_{\mathrm{i}}' = 4\pi G\rho_0 r\left\{\left[-\frac{1}{3}+\frac{2}{5}s_2(P_2+3\beta\sin\theta\cos\theta\sin(\alpha-\varphi))\right]\mathbf{e}_r + \frac{3}{5}s_2\left[\cos\theta\sin\theta-\beta\left(\cos^2\theta-\sin^2\theta\right)\sin(\alpha-\varphi)\right]\mathbf{e}_\theta + \frac{3}{5}s_2\beta\cos\theta\cos(\alpha-\varphi)\mathbf{e}_\varphi\right\} \quad (30)$$

and

$$-\boldsymbol{\nabla}\phi_{\mathrm{e}}' = 4\pi G\rho_0 \times\frac{R^5}{r^4}\left\{\left[-\frac{r^2}{3R^2}-\frac{3}{5}s_2(P_2+3\beta\sin\theta\cos\theta\sin(\alpha-\varphi))\right]\mathbf{e}_r + \frac{3}{5}s_2\left[\cos\theta\sin\theta-\beta\left(\cos^2\theta-\sin^2\theta\right)\sin(\alpha-\varphi)\right]\mathbf{e}_\theta + \frac{3}{5}s_2\beta\cos\theta\cos(\alpha-\varphi)\mathbf{e}_\varphi\right\}. \quad (31)$$

Thus, the variation of the radial component of the gravity,

$$-\frac{\partial\phi_1}{\partial r} = -\frac{\partial\phi'}{\partial r}+\frac{\partial\phi}{\partial r},$$

between the reference position and the tilted position is

$$-\frac{\partial\phi_{1\mathrm{i}}}{\partial r} = 4\pi G\rho_0\left[\frac{2r}{5}s_2 3\beta\cos\theta\sin\theta\sin(\alpha-\varphi)\right] \quad (32)$$

and

$$-\frac{\partial\phi_{1\mathrm{e}}}{\partial r} = 4\pi G\rho_0\left[-\frac{3R^5}{5r^4}s_2 3\beta\cos\theta\sin\theta\sin(\alpha-\varphi)\right]. \quad (33)$$

Since these variations are of first order in s_2, the jump of radial gravity is, to first order in the flattening, the same at the spherical surface, $r=R$, and at the spheroidal surface (eq. 4):

$$\frac{\partial\phi_{1\mathrm{i}}}{\partial r}+4\pi G\rho_0[R\, s_2 3\beta\cos\theta\sin\theta\sin(\alpha-\varphi)] = \frac{\partial\phi_{1\mathrm{e}}}{\partial r}. \quad (34)$$

The latter expression can be rewritten as

$$\frac{\partial\phi_{1\mathrm{i}}}{\partial r}+\frac{3}{R}\phi_{1\mathrm{i}} = -4\pi G\rho_0[R\, s_2 3\beta\cos\theta\sin\theta\sin(\alpha-\varphi)]. \quad (35)$$

3 COMPUTATION OF A RIGID NUTATION BASED ON THE DEFORMATION EQUATIONS

3.1 Summary of the method

Smith (1974) developed a theory for calculating the normal modes and the forced motions of a deformable, slightly ellipsoidal, rotating body, described by an isotropic elastic constitutive relation. Here, our goal is to rederive the results obtained in the previous section using Smith's method. However, beforehand we must be sure that a rigid nutation of a *deformable* body, in a given reference frame, is indeed a possible motion. Obviously, this is the case because the particular motion considered corresponds to a free mode usually called the *tilt-over mode* (TOM). The latter exists for every rotating body. Viewed in an inertial reference frame, the TOM is merely the steady rigid rotation of the body about an axis that does not coincide with any axis of the relative reference system O*xyz* described above. Thus, its period is the spin period of the body, namely one sidereal day for the Earth. It does not depend on the internal constitution of the body (Smith 1977; Moritz & Mueller 1987).

We briefly describe the main steps of the method elaborated by Smith (1974), adopting throughout this section the notation of Smith's paper and matching it with the notation used earlier. The reference body is assumed to be in hydrostatic equilibrium and slowly rotating. It is, therefore, slightly flattened into a spheroidal shape. Because, as a rule, earth models are tabulated as spherically symmetric models, we must start by computing the flattening $\varepsilon(r)$ that the internal strata would assume if the models were steadily rotating about a fixed axis. Various methods can be used to perform this task. The most efficient ones are explained in detail in the paper by Denis *et al.* (1998). The density at a geometrical point (r, θ, φ) of the spheroidal body can then be expressed as a function of the density $\rho_0(r)$ *at the same geometrical point* of the corresponding non-rotating spherical body by the formula

$$\rho(r, \theta, \varphi) = \rho_0(r)+\rho_2(r)\, P_2(\cos\theta), \quad (36)$$

where

$$\rho_2(r) = -s_2(r)\, r\,\frac{d\rho_0}{dr}. \quad (37)$$

The two elastic parameters can be split in the same way, whilst the gravity potential is

$$\phi(r, \theta, \varphi) = \phi_0(r)-\left[s_2(r)\, r\,\frac{d\phi_0}{dr}+\frac{\Omega_0^2}{3}r^2\right]P_2(\cos\theta), \quad (38)$$

Ω_0 being the angular speed of rotation of the reference frame.

After linearizing and Fourier-transforming the equations of motion, they can be written as a set of scalar differential equations in the same way as is done for the deformation equations of a spherical model, that is, the vector fields are decomposed into toroidal and spheroidal parts described in terms of scalars. Each scalar field is then represented as a series of surface spherical harmonics. In practice, generalized surface spherical harmonics are used instead of common spherical harmonics, but this is a somewhat technical point.

In the spherical, non-rotating case, the spheroidal and toroidal deformations are uncoupled from each other, as are the different harmonic degrees, and, for a given degree, the order is irrelevant. In the rotating spherical and/or ellipsoidal case, owing to expressions such as eq. (36), to the Coriolis force and to the centrifugal force, the presence of products of spherical harmonics couples the spheroidal and toroidal motions. However, since a spheroidal model has axial symmetry, there is no coupling between deformation of different harmonic orders, and a spheroidal deformation of odd (even) degree is coupled to toroidal deformations of even (odd) degrees and to spheroidal deformations of odd (even) degrees. To solve the equations of motion effectively, the coupling chain must be truncated at some level. Nobody has ever proved that such a truncation procedure is valid and leads to correct results, but we assume that it is in the circumstances we are interested in. An additional approximation introduced by Smith is to neglect the terms of orders higher than first in the flattening.

The second important step of the theory, after separating the unperturbed density, elastic parameters and gravity potential into a sum of terms of harmonic degree 0 and a term of harmonic degree 2 and order 0, consists of writing down the continuity and boundary conditions of the perturbed quantities and the displacement. These conditions obviously apply at the spheroidal surfaces. The trick here is to expand each of the perturbed quantities, displacement and unit normal with respect to the spheroidal surface into a Taylor series in the neighbourhood of the corresponding equivolumetric sphere of radius r and to neglect the terms of order higher than 1 in the flattening. If we put

$$h(r,\theta)=s_2 rP_2=h_2P_2 \tag{39}$$

to reconcile Smith's notation with ours, the normal to an ellipsoidal surface is

$$\mathbf{n}=\mathbf{e}_r-\frac{1}{r}\frac{\partial h}{\partial r}\,\mathbf{e}_\theta\,. \tag{40}$$

For example, consider a simple interface described by the position vector

$$\mathbf{p}=\mathbf{r}+h_2P_2\mathbf{e}_r\,. \tag{41}$$

The condition that this interface is welded (solid–solid) is expressed by the condition that the displacement vector

$$\mathbf{u}(\mathbf{r})+h_2P_2\,\frac{\partial \mathbf{u}}{\partial r} \tag{42}$$

must be continuous at $\mathbf{r}$. According to Smith, only terms of first order in the flattening ought to be retained in this condition, as well as in the other transformed continuity conditions and in the linearized equations of motion. This set of approximate equations and conditions would then give the same results as the exact equations of motion and continuity conditions. I will now show that this approximation is not a good one under certain circumstances. The reason for this is in fact quite simple: for certain motions, such as the TOM or the free core nutation, the terms of order 0 in the flattening (that is, terms that do not depend on the flattening) are smaller than or of the same amplitude as terms of order 1. It thus becomes necessary to take into account terms of order 2 in the flattening to obtain acceptable numerical results.

3.2 The TOM as a solution of the deformation equations

Knowing a particular proper mode specific to a rotating body, i.e. the TOM, we want to verify that the latter is indeed a solution of the equations of motion and satisfies the appropriate boundary conditions. Since the TOM is a pure rotation without deformation, although it involves a variation of gravity, we may keep only two terms amongst the infinite number of the chain, namely

$$\mathbf{u}=\boldsymbol{\tau}_1^1+\boldsymbol{\sigma}_2^1\,, \tag{43}$$

where $\boldsymbol{\tau}_1^1$ denotes the degree 1 and order 1 toroidal displacement, and $\boldsymbol{\sigma}_2^1$ denotes the degree 2 and order 1 spheroidal displacement. The second term, $\boldsymbol{\sigma}_2^1$, is exactly zero, but we keep it to remind us that a degree 2 and order 1 perturbation of the gravity potential is involved. The spheroidal scalars, that is, the radial displacement U_2^1 and the scalar V_2^1 proportional to the displacement tangential to the equivolumetric sphere, are thus null. Because there is no deformation at all, the elastic stress tensor must also be zero. This means that the spheroidal and toroidal parts of the normal stress, and their derivatives, vanish:

$$P_2^1=0\,,\quad Q_2^1=0\,,\quad R_1^1=0\,. \tag{44}$$

Finally, we know that the period of the TOM is exactly one sidereal day, corresponding to an angular frequency

$$\omega=\Omega_0=7.292115\times10^{-5}\ \mathrm{rad\ s^{-1}}\,. \tag{45}$$

3.2.1 *The TOM of a homogeneous body*

Let us first consider a homogeneous body. Then, ρ_0 is constant and h_2 and ϕ_2 are

$$h_2=\frac{-5\Omega_0^2 r}{8\pi G\rho_0} \tag{46}$$

and

$$\phi_2=-h_2\,\frac{d\phi_0}{dr}=\frac{5}{6}\,\Omega_0^2r^2\,. \tag{47}$$

In this special case, as $\rho_2=0$, we can use the equations of motion given by Smith (1974). His three scalar equations, (5.28), (5.29) and (5.30), then reduce to

$$\phi_{1\,2}^{\,1}=\sqrt{3}W_1^1\left(\frac{1}{6}\,\Omega_0^2r-\frac{\phi_2}{2r}\right), \tag{48}$$

where W_1^1 is the degree 1 and order 1 toroidal scalar of the displacement, and $\phi_{1\,2}^{\,1}$ is the radial part of the degree 2 and order 1 variation of the gravity potential. $\phi_{1\,2}^{\,1}$ obeys the Laplace equation,

$$\frac{d^2\phi_{1\,2}^{\,1}}{dr^2}+\frac{2}{r}\frac{d\phi_{1\,2}^{\,1}}{dr}-\frac{6\phi_{1\,2}^{\,1}}{r^2}=0\,. \tag{49}$$

The solution of eq. (49), which is regular at the origin, is proportional to r^2. This is consistent with eq. (48) if W_1^1 is a linear function of r, that is, the motion is a rigid rotation about an axis lying in the equatorial plane:

$$\begin{aligned}\phi_{1\,2}^{\,1}&=-\frac{\sqrt{3}}{4}\,W_1^1\Omega_0^2r\\&=\frac{2\sqrt{3}}{5}\,W_1^1\pi G\rho_0 s_2 r\,.\end{aligned} \tag{50}$$

That W_1^1 must be a linear function of r is also consistent with equation (5.24) of Smith, which gives the radial derivative of W_1^1. By replacing $\sin(\alpha-\varphi)$ with $e^{i(\alpha-\varphi)}$ in eq. (28) and calculating the Fourier transform, we obtain, if $\alpha=-\Omega_0 t$ and $W_1^1=4\pi\sqrt{2}\,r\,\beta$, the solution (50) multiplied by the generalized spherical harmonic $\mathscr{D}_{10}^2$ and by the delta function $\delta(\omega-\Omega_0)$. The factor $\sqrt{2}$ in W_1^1 stems from the definition of the generalized spherical harmonics,

$$\mathscr{D}_{10}^2(\theta,\varphi)=\frac{\sqrt{6}}{2}\cos\theta\sin\theta\,e^{i\varphi}\,, \tag{51}$$

$$\mathscr{D}_{10}^1(\theta,\varphi)=\frac{\sqrt{2}}{2}\sin\theta\,e^{i\varphi}\,. \tag{52}$$

We can now look at the boundary conditions. Of course, at the external boundary,

$$\mathbf{p}_R=[R+h_2(R)P_2]\mathbf{e}_r\,, \tag{53}$$

we must also find the relation (48). The continuity conditions were written down by Smith, but not the boundary conditions. Consequently, we shall first derive the latter for an arbitrary movement. As usual, the variation ϕ_1 of the gravitational potential must be continuous at the external boundary, and its gradient must obey the relation

$$(\boldsymbol{\nabla}\phi_{1\mathrm{i}}+4\pi\mathrm{G}\rho\mathbf{u})\cdot\mathbf{n}=\boldsymbol{\nabla}\phi_{1\mathrm{e}}\cdot\mathbf{n}\,. \tag{54}$$

Using eq. (40), expanding $\phi_1(\mathbf{p}_R)$ and $\boldsymbol{\nabla}\phi_1(\mathbf{p}_R)$ into Taylor series, and noting that

$$\rho(\mathbf{p}_R)=\rho_0(R\mathbf{e}_r)\,, \tag{55}$$

we find to first order in h_2

$$\frac{\partial\phi_{1\mathrm{i}}}{\partial r}+4\pi\mathrm{G}(\rho_0+\rho_2P_2)\mathbf{u}\cdot\mathbf{e}_r+\frac{\partial}{\partial r}\left(h_2P_2\,\frac{\partial\phi_{1\mathrm{i}}}{\partial r}+4\pi\mathrm{G}\rho_0\mathbf{u}\cdot\mathbf{e}_r\right)$$
$$-\boldsymbol{\nabla}(h_2P_2)\cdot(\boldsymbol{\nabla}\phi_{1\mathrm{i}}+4\pi\mathrm{G}\rho_0\mathbf{u})$$
$$=\frac{\partial\phi_{1\mathrm{e}}}{\partial r}+\frac{\partial}{\partial r}\left(h_2P_2\,\frac{\partial\phi_{1\mathrm{e}}}{\partial r}\right)-\boldsymbol{\nabla}(h_2P_2)\cdot\boldsymbol{\nabla}\phi_{1\mathrm{e}}\,. \tag{56}$$

We skip the detailed and tedious expansion of this relation into generalized spherical harmonics. The result for the left-hand side is

$$\frac{d\phi_{1\mathrm{i}}{}_{\ell}^{m}}{dr}+4\pi\mathrm{G}\rho_0U_\ell^m+h_2\sum_{\ell'=|\ell-2|}^{|\ell+2|}\left\{\begin{bmatrix}\ell&2&\ell'\\0&0&0\\m&0&m\end{bmatrix}\right.$$
$$\times\left[4\pi\mathrm{G}\left(-\sqrt{\frac{\ell'(\ell'+1)}{2}}\,\frac{\rho_0}{r}\,V_{\ell'}^m+\left(\frac{2\rho_0}{r}-\frac{d\rho_0}{dr}\right)U_{\ell'}^m\right)\right.$$
$$\left.-\frac{2}{r}\frac{d\phi_{1\mathrm{i}}{}_{\ell'}^{m}}{dr}+\frac{\ell'(\ell'+1)}{r^2}\,\phi_{1\mathrm{i}}{}_{\ell'}^{m}\right]+\begin{bmatrix}\ell&2&\ell'\\0&1&-1\\m&0&m\end{bmatrix}\frac{\sqrt{3}}{r}$$
$$\left.\times\left[\frac{\phi_{1\mathrm{i}}{}_{\ell'}^{m}}{r}\sqrt{\frac{\ell'(\ell'+1)}{2}}\left(1+(-1)^{\ell+\ell'}\right)+4\pi\mathrm{G}\rho_0\begin{bmatrix}-V_{\ell'}^m\\W_{\ell'}^m\end{bmatrix}\right]\right\}. \tag{57}$$

The brackets

$$\begin{bmatrix}-V_{\ell'}^m\\W_{\ell'}^m\end{bmatrix}$$

mean that we must take $-V_{\ell'}^m$ if $\ell+\ell'$ is even and $W_{\ell'}^m$ if $\ell+\ell'$ is odd, whereas the arrays such as

$$\begin{bmatrix}\ell&2&\ell'\\0&0&0\\m&0&m\end{bmatrix}$$

designate products of Wigner 3-j symbols, as defined by Smith (1974). These coefficients come from products of generalized spherical harmonics $\mathscr{D}_{mN}^{\ell}$. At present, we know that ϕ_{1_e} is a solution of the Laplace equation. Therefore, we may write

$$\phi_{1\mathrm{e}}=\sum_{\ell=0}^{\infty}\frac{B_\ell}{r^{\ell+1}}\,\mathscr{D}_{m0}^{\ell}\,, \tag{58}$$

where the B_ℓ are constants. If we substitute this formula into the boundary condition expressing the continuity of the variation of the gravity potential,

$$\phi_{1\mathrm{i}}{}_{\ell}^{m}+\sum_{\ell'=|\ell-2|}^{|\ell+2|}\begin{bmatrix}\ell&2&\ell'\\0&0&0\\m&0&m\end{bmatrix}h_2\,\frac{d\phi_{1\mathrm{i}}{}_{\ell'}^{m}}{dr}$$
$$=\phi_{1\mathrm{e}}{}_{\ell}^{m}+\sum_{\ell'=|\ell-2|}^{|\ell+2|}\begin{bmatrix}\ell&2&\ell'\\0&0&0\\m&0&m\end{bmatrix}h_2\,\frac{d\phi_{1\mathrm{e}}{}_{\ell'}^{m}}{dr}\,, \tag{59}$$

we obtain an algebraic system whose unknowns are the coefficients B_ℓ. If we denote the left-hand side of eq. (59) by $\boldsymbol{\Phi}_\ell$, we can write that system under the form $\boldsymbol{\Phi}=\mathbf{AB}$, where $\mathbf{B}$ and $\boldsymbol{\Phi}$ are the column vectors

$$\mathbf{B}=(\ldots B_{\ell-4},\,B_{\ell-2},\,B_\ell,\,B_{\ell+2},\,B_{\ell+4},\,\ldots)$$

and

$$\boldsymbol{\Phi}=(\ldots\boldsymbol{\Phi}_{\ell-4},\,\boldsymbol{\Phi}_{\ell-2},\,\boldsymbol{\Phi}_\ell,\,\boldsymbol{\Phi}_{\ell+2},\,\boldsymbol{\Phi}_{\ell+4},\,\ldots)\,.$$

The matrix $\mathbf{A}$ is tridiagonal with off-diagonal elements of first order in h_2 and diagonal elements given by

$$\frac{1}{r^{k+1}}\left\{1-(k+1)\,\frac{h_2}{r}\begin{bmatrix}k&2&k\\0&0&0\\m&0&m\end{bmatrix}\right\}, \tag{60}$$

where k can take the values $\ldots\ \ell-4,\ell-2,\ell,\ell+2,\ell+4,\ \ldots$. Still neglecting terms of order higher than first in h_2, we easily obtain the solution

$$B_k=\boldsymbol{\Phi}_kr^{k+1}\left\{1+(k+1)\frac{h_2}{r}\begin{bmatrix}k&2&k\\0&0&0\\m&0&m\end{bmatrix}\right\}$$
$$+\boldsymbol{\Phi}_{k-2}h_2r^k(k-1)\begin{bmatrix}k&2&k-2\\0&0&0\\m&0&m\end{bmatrix}$$
$$+\boldsymbol{\Phi}_{k+2}h_2r^k(k+3)\begin{bmatrix}i&2&k+2\\0&0&0\\m&0&m\end{bmatrix}. \tag{61}$$

The boundary condition for gravity finally becomes

$$\frac{d\phi_{1i\ell}^{\ m}}{dr}+4\pi G\rho_0 U_\ell^m+\frac{(\ell+1)}{r}\,\phi_{1i\ell}^{\ m}$$
$$+h_2\sum_{\ell'=|\ell-2|}^{|\ell+2|}\left\{\begin{bmatrix}\ell & 2 & \ell'\\ 0 & 0 & 0\\ m & 0 & m\end{bmatrix}\left[4\pi G\left(-\sqrt{\frac{\ell'(\ell'+1)}{2}}\,\frac{\rho_0}{r}\,V_{\ell'}^m\right.\right.\right.$$
$$\left.\left.+\left(\frac{2\rho_0}{r}-\frac{d\rho_0}{dr}\right)U_{\ell'}^m\right)+\frac{(\ell-1)}{r}\,\frac{d\phi_{1i\ell'}^{\ m}}{dr}+\frac{(\ell-1)(\ell'+1)}{r^2}\,\phi_{1i\ell'}^{\ m}\right]$$
$$\left.+\begin{bmatrix}\ell & 2 & \ell'\\ 0 & 1 & -1\\ m & 0 & m\end{bmatrix}4\pi G\rho_0\frac{\sqrt{3}}{r}\begin{bmatrix}-V_{\ell'}^m\\ W_{\ell'}^m\end{bmatrix}\right\}=0\,. \tag{62}$$

We now go back to our main task, which is the derivation of the boundary conditions applying for the TOM. Noting that

$$\begin{bmatrix}\ell & 2 & \ell-1\\ 0 & 1 & -1\\ m & 0 & m\end{bmatrix}=-\frac{m}{2\ell-1}\sqrt{\frac{3(\ell-m)(\ell+m)}{2\ell(\ell-1)}}\,, \tag{63}$$

it is not very complicated to see that the condition (62) yields, to first order in h_2, the first equality of eq. (50).

The other three boundary conditions state that the components of the traction vanish at the surface of the homogeneous ellipsoid. We consider here the radial component P_2^1, but our reasoning would be the same for the tangential components. Thus, at the ellipsoidal boundary, the continuity condition (5.44) of Smith (1974) gives

$$P_2^1+h_2\left(\frac{1}{7}\frac{dP_2^1}{dr}+\frac{\sqrt{3}}{14}\frac{Q_2^1}{r}-\frac{\sqrt{3}}{2}\frac{R_1^1}{r}\right)=0\,. \tag{64}$$

Of course, P_2^1, its derivative, Q_2^1 and R_1^1 are exactly zero, because there is no deformation. However, it can happen when we solve the problem numerically that on substituting in $(dP_2^1)/dr$—its expression provided by the radial part of the equation of motion—we *must* keep terms of first order in h_2, and thus terms of second order in the constraint equation (64). Indeed, the equation of motion (5.28) of Smith (1974) yields

$$\frac{dP_2^1}{dr}=\rho_0\left(\frac{d\phi_{1}{}_2^1}{dr}+\frac{d\phi_2}{dr}\frac{\sqrt{3}}{2}\frac{W_1^1}{r}-\frac{\sqrt{3}}{3}\Omega_0^2W_1^1\right), \tag{65}$$

which obviously agrees with eq. (48) if $(dP_2^1)/dr=0$. This shows that if we retain only the first-order terms in eq. (64), we would *not* recover eq. (50). Thus, I have found numerically, for a homogeneous model of density $\rho_0=5515$ kg m^{-3}, a period of 87 109 s when the terms of second order are neglected. This value disagrees significantly with the period of the TOM ($T_{\rm TOM}=86$ 164.1 s). Including the second-order terms, I find a period equal to 86 164.33 s. Of course, from a numerical point of view, it would be more practical to replace directly in eq. (64) the value of $(dP_2^1)/dr$ computed from the equations of motion. In the following, we show that the second-order terms also ought to be kept in the equations of motion of a heterogeneous ellipsoid.

3.2.2 *The TOM of a heterogeneous body*

For a heterogeneous body, the density and flattening are no longer constants, but vary along the radius r. The function $\varepsilon(r)$ can be obtained, for example, by solving Clairaut's equation (Denis *et al.* 1998). Denoting the Cauchy stress tensor by $\boldsymbol{T}^e$, the equation of motion in vector form can then be written as

$$\boldsymbol{\nabla}\cdot\boldsymbol{T}^e=(\rho_0+\rho_2P_2)(\boldsymbol{\eta}_1+\boldsymbol{\eta}_2+\boldsymbol{\eta}_3)\,, \tag{66}$$

where

$$\boldsymbol{\eta}_1=-\omega^2\mathbf{u}+\boldsymbol{\nabla}\phi_1-(\boldsymbol{\nabla}\cdot\mathbf{u})\left(\boldsymbol{\nabla}\phi_0-\frac{\Omega_0^2}{3}\,r\mathbf{e}_r\right)$$
$$+\boldsymbol{\nabla}\left[\mathbf{u}\cdot\left(\boldsymbol{\nabla}\phi_0-\frac{\Omega_0^2}{3}\,r\mathbf{e}_r\right)\right], \tag{67}$$

$$\boldsymbol{\eta}_2=2i\omega\boldsymbol{\Omega}_0\times\mathbf{u} \tag{68}$$

and

$$\boldsymbol{\eta}_3=\boldsymbol{\nabla}[\mathbf{u}\cdot\boldsymbol{\nabla}(\phi_2P_2)]-(\boldsymbol{\nabla}\cdot\mathbf{u})\boldsymbol{\nabla}(\phi_2P_2)\,. \tag{69}$$

In his equation (5.27a), Smith assumes that the term $\rho_2\,P_2\,\boldsymbol{\eta}_3$ is negligible. (We note that Smith's vector $\boldsymbol{\eta}_3$ is our vector $\boldsymbol{\eta}_3$ multiplied by ρ_0). However, this term is certainly not negligible, since, in the case of the TOM, our equation of motion yields the following scalar equations:

$$\frac{dP_2^1}{dr}=0=\left(\rho_0+\frac{1}{7}\rho_2\right)\left(\frac{d\phi_{1}{}_2^1}{dr}+\frac{d\phi_2}{dr}\frac{\sqrt{3}}{2}\frac{W_1^1}{r}-\frac{\sqrt{3}}{3}\Omega_0^2W_1^1\right), \tag{70}$$

$$\frac{dQ_2^1}{dr}=0=\left(\rho_0+\frac{1}{14}\rho_2\right)\left(-\frac{2\sqrt{3}}{r}\phi_{1}{}_2^1-\frac{3\phi_2}{r}\frac{W_1^1}{r}+\Omega_0^2W_1^1\right) \tag{71}$$

and

$$\frac{dR_1^1}{dr}=0=\frac{3}{10}\rho_2\left(-\frac{2\sqrt{3}}{r}\phi_{1}{}_2^1-\frac{3\phi_2}{r}\frac{W_1^1}{r}+\Omega_0^2W_1^1\right), \tag{72}$$

from which the solution (48) can again be deduced. As ϕ_2 obeys

$$\frac{d^2\phi_2}{dr^2}+\frac{2}{r}\frac{d\phi_2}{dr}-\frac{6\phi_2}{r^2}=4\pi G\rho_2\,, \tag{73}$$

it can be shown that the solution (48) is also a solution of the Poisson equation,

$$\frac{d^2\phi_{1}{}_2^1}{dr^2}+\frac{2}{r}\frac{d\phi_{1}{}_2^1}{dr}-\frac{6\phi_{1}{}_2^1}{r^2}=-4\pi G\rho_2\frac{\sqrt{3}}{2}\frac{W_1^1}{r}\,. \tag{74}$$

In view of (48), the reason why the second-order terms are not negligible now becomes evident: whereas the spheroidal part of the displacement vector vanishes, the potential perturbation is proportional to the flattening and to the toroidal displacement of degree 1. In the equation of motion (66), $\boldsymbol{\eta}_3$ is thus of the same order as $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$, i.e. of first order in ellipticity. Obviously, the second-order terms must also be kept in the continuity conditions.

Numerically, for earth model PREM (Dziewonski & Anderson 1981), I obtain a period of 86 164.22 s. The discrepancy between this period and the sidereal day is only 0.12 s. Moreover, the spheroidal scalars U_2^1 and V_2^1 are found to be, at most, of the order of 10^{-6} W_1^1. The function $W_1^1(r)$ is plotted in Fig. 1. The plot clearly represents a rigid rotation of the whole body. Without considering the second-order terms in ellipticity in the equations of motion and in the continuity conditions, I obtain a period of 86 703.74 s and a function $W_1^1(r)$ showing three different rigid rotations for the mantle and the inner and outer cores.

In Smith (1977) and Smith & Dahlen (1981), the only second-order terms that are not neglected are the second-order centrifugal terms. They stem from the product between the latitude-dependent part of the density, $\rho_2 P_2$, and the centrifugal potential contained in the vector $\boldsymbol{\eta}_1$. However, their influence is very small because they only involve the spheroidal scalars U_ℓ^1 and V_ℓ^1, which are much smaller than the dominant toroidal scalar W_1^1 in nutational motions.

Schastok (1997) went one step further: he developed ρ_2, ϕ_2 and h_2 up to second order in ε, and some of the continuity conditions up to second order in h_2. He also added a fourth-degree harmonic term, h_4, proportional to ε^2, in the equations of the boundary surfaces. Although it is fully justified to include in the theory *all* of the terms of second order in the ellipticity, and it thus seems necessary to consider a second-order hydrostatic theory as explained e.g. in Denis *et al.* (1998), Schastok did not achieve a better precision than us regarding the period of the TOM. To obtain for this particular mode a period of exactly one sidereal day, he adjusted the ellipticity of the earth model in a somewhat arbitrary way, without altering the density of the model. Such a procedure is inconsistent with hydrostatic theory (Smith & Dahlen 1981).

We can now turn to other free motions of an ellipsoidal earth characterized by a large degree 1 toroidal displacement of some of its layers, associated with much smaller deformation, namely free core nutation and free inner core nutation.

4 FREE CORE NUTATION AND FREE INNER CORE NUTATION

Smith (1977) applied his method to investigate the free modes of rotating, slightly ellipsoidal earth models. Besides the successful computation of the periods of the Chandler wobble of the mantle and the free core nutation (FCN, also called the nearly diurnal free wobble, NDFW), the computation of the nutation or the Chandler wobble of the solid inner core was problematic and the inner core showed an anomalous behaviour for the TOM, the FCN and the Chandler wobble of the mantle. The arguments used by Smith (1977) to explain this failure of the method were of two kinds. First, Smith stated that the anomalous behaviour of the inner core during the diurnal or nearly diurnal motions were in part due to the limited precision of the numerical algorithm. Moreover, Smith gave a second argument to explain the unexpectedly large rigid rotation that he found of the inner core during the Chandler wobble of the mantle. This argument was based on a paper by Busse (1970), who considered a simple non-gravitating model with a rigid oblate inner core undergoing a Chandler wobble. Busse showed that the associated motion occurring in the liquid, incompressible, non-viscous core surrounding the inner core and enclosed within a rigid oblate mantle at rest vanishes outside the cylinder with axis parallel to the initial rotation axis and a radius equal to the semi-major axis of the inner core. Thus, the almost discontinuous displacement within the fluid could not be approximated by means of the truncated series

$$\mathbf{u} = \boldsymbol{\tau}_1^{\pm 1} + \boldsymbol{\sigma}_2^{\pm 1} + \boldsymbol{\tau}_3^{\pm 1} \,. \tag{75}$$

However, as second-order terms in the ellipticity cannot be neglected, we have adopted a technique different from that of Smith (1977) when applying the continuity conditions and propagating the numerical solutions across boundaries where the density and/or the Lamé parameters are discontinuous. This is especially the case at the ICB and the CMB. If the

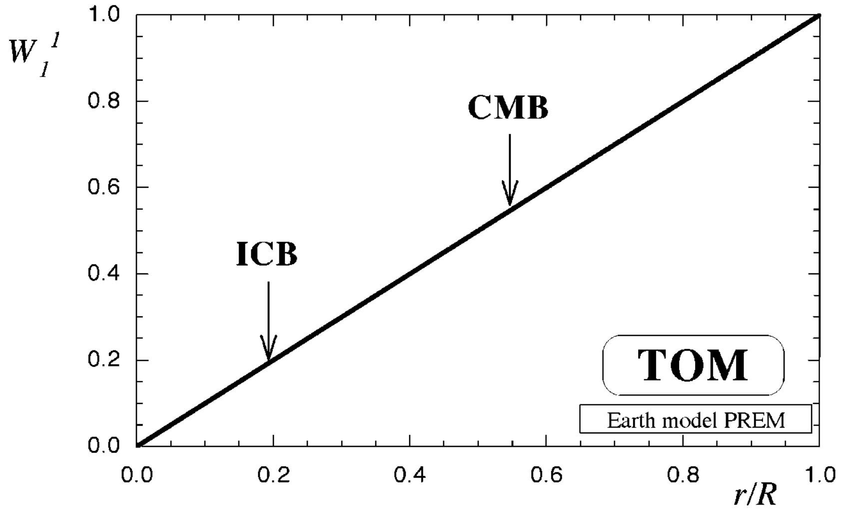

Figure 1. Toroidal displacement scalar W_1^1 of degree 1 and order 1 for the tilt-over mode (TOM) of an oceanless version of PREM as a function of the fractional radius r/R. The normalization of the eigenfunction is such that $W_1^1 = 1$ at $r = R$. A linear regression shows that the non-linear part of W_1^1, which theoretically should be strictly zero, is of the order of $10^{-6}\,|W_1^1|$, and the spheroidal scalars U_2^1 and V_2^1 are of the same magnitude or even smaller. This shows that the overall numerical accuracy is about 10^{-5}. The computed eigenperiod is 86 164.22 s. The locations of the inner core (ICB) and core–mantle (CMB) boundaries are indicated. Because W_1^1 is a linear function of the radius, the motion consists of a rigid rotation of the whole body about an axis lying in the equatorial plane.

displacement vector is assumed to be sufficiently well approximated by the truncated series (75), the system of differential equations in a solid part of the earth is of order 10. Starting near the centre, we must consequently propagate through the inner core five linearly independent vectors $y^{(I)}, \ldots, y^{(V)}$, which are regular at $r=0$. The 10 components of the vectors y are the spheroidal and toroidal components of the displacement and of the traction, the potential perturbation $\phi_{1\,2}^{\pm 1}$ and the scalar

$$\frac{d\phi_{1\,2}^{\pm 1}}{dr} + 4\pi G\rho_0 U_2^{\pm 1}.$$

In the liquid outer core, the differential system is of order 4 only. There we propagate four linearly independent vectors, $\tilde{y}^{(I)}, \ldots, \tilde{y}^{(IV)}$, from the top side of the ICB up to the bottom side of the CMB. The vectors $\tilde{y}$ have seven components, the tangential components of the traction being zero in a non-viscous fluid. Next, we integrate 10 linearly independent vectors from the top of the CMB to the free surface, the propagation of the solution vectors through solid–solid interfaces posing no real problem. (We do not consider earth models with a global ocean or we modify the existing models to replace the global ocean with a solid crust, but the presence of a liquid outer shell could be treated in the same way as the liquid outer core.) Finally, the vanishing of the determinant of the 19×19 matrix built on the continuity conditions at the ICB and the CMB and on the boundary conditions at the free surface will occur when the frequency of the motion is an eigenfrequency of the model.

Some results of our computations for PREM are shown in Figs 2–5 and in Table 1. Fig. 2 displays the scalar of the toroidal displacement field of degree 1 and order 1 relative to the FCN. The corresponding calculated period is -458.6 sidereal days. The normalization of the eigenfunctions is such that $W_1^1(R)=1$. It is seen that the inner and outer cores rotate in the same sense whereas the mantle rotates in the opposite sense. Table 1 compares our results with the corresponding results of Mathews *et al.* (1991b) and de Vries & Wahr (1991). Both studies extended the semi-analytical computation of the free and forced nutations of the Earth developed by Sasao *et al.* (1980) to take into account the presence of a solid inner core. By means of a linear regression, we have estimated the linear part of W_1^1 in each shell. The ratio between the amplitudes of the rigid rotations of the inner core and of the mantle is designated by $\tilde{\zeta}_s/\tilde{\zeta}$; the corresponding ratio for the outer core is denoted by $\tilde{\zeta}_f/\tilde{\zeta}$.

The methods of Sasao *et al.* (1980), Mathews *et al.* (1991a) and de Vries & Wahr (1991) rely on the calculation of *static* Love numbers of a spherical model. However, Denis *et al.* (1998) have demonstrated that for realistic earth models, static

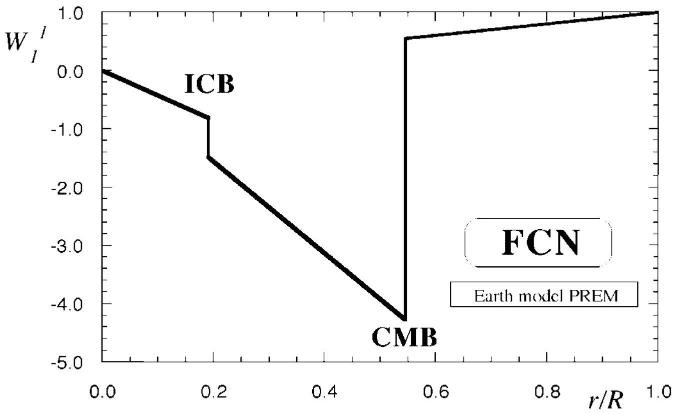

Figure 2. Same as Fig. 1 but for FCN. The period is -458.6 sidereal days. Note that the inner core and the outer core rotate in the same sense while the mantle rotates in the opposite sense.

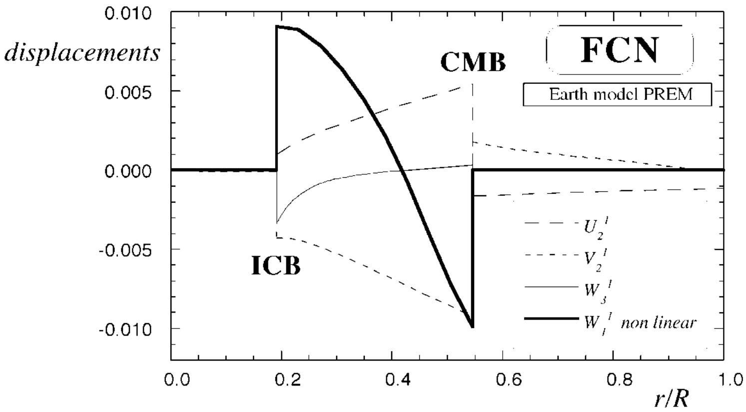

Figure 3. The heavy solid line represents the non-linear part of the degree 1 and order 1 toroidal scalar W_1^1 of the FCN for an oceanless PREM model as a function of the fractional radius. It was obtained by subtracting the linear fit of W_1^1 from W_1^1 separately in the inner core, in the outer core and in the mantle. The dashed lines are the degree 2 and order 1 spheroidal scalars, and the light solid line is the degree 3 and order 1 toroidal scalar describing the displacements involved in the FCN.

Table 1. Periods (in *sidereal days*) of the FCN and FICN, and the corresponding relative rotation angles between the outer or inner core and the mantle. The negative (resp. positive) sign of an eigenperiod reflects the retrograde (resp. prograde) motion of the rotation axis in space. The model of reference is an oceanless version of PREM. The ratios $\tilde{\zeta}_s/\tilde{\zeta}$ and $\tilde{\zeta}_f/\tilde{\zeta}$ are not given explicitly in the paper of de Vries & Wahr (1991), but can be deduced from other results provided.

	FCN			FICN		
	Period	$\tilde{\zeta}_f/\tilde{\zeta}$	$\tilde{\zeta}_s/\tilde{\zeta}$	Period	$\tilde{\zeta}_f/\tilde{\zeta}$	$\tilde{\zeta}_s/\tilde{\zeta}$
Mathews *et al.* (1991b)	−457.04	−7.821	−4.393	476.8	12.40	−3139
de Vries & Wahr (1991)	−457	−7.835	−4.287	471	11.69	−3017
This paper	−458.6	−7.83	−4.27	473.9	11.4	−3025

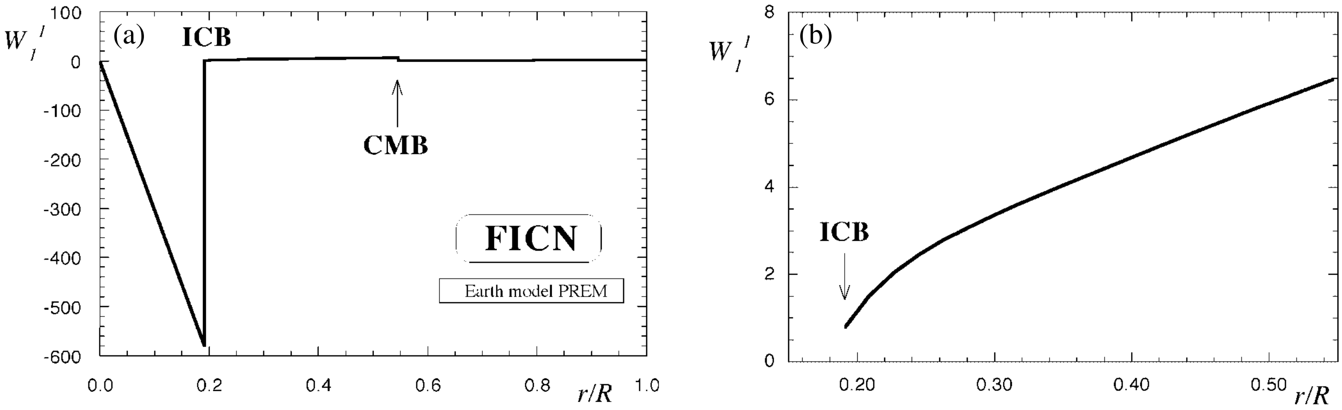

Figure 4. (a) Same as Fig. 1 for the FICN (period 473.9 sidereal days). There is almost no rotation in the outer core and in the mantle. (b) W_1^1 in the outer core drawn at a smaller scale.

Love numbers *sensu stricto* do not exist and, therefore, such a calculation is *a priori* meaningless. Nevertheless, we assume that, in this case, the physically incorrect numerical trick necessary to overcome the problem of static deformations, implying discontinuity of the radial displacement at the CMB (e.g. Smylie & Mansinha 1971; Chinnery 1975), leads to numerically acceptable results for the nutations. In order to compare the amplitude of the deformation associated with the nutation, the non-linear part of W_1^1 is plotted in Fig. 3 with the degree 2 spheroidal and degree 3 toroidal scalars of the displacement fields. The discrepancy between the computed FCN period and that deduced from the observations, i.e. about 431.2 sidereal days, has been tentatively explained by Gwinn *et al.* (1986) and Neuberg *et al.* (1987) and more recently Dehant & Defraigne (1997) and Schastok (1997) by considering the non-hydrostatic structure of the earth.

We now look at the free inner core nutation, or FICN. Fig. 4 shows the W_1^1 function corresponding to an eigenperiod of 473.9 sidereal days. The graph obviously represents a free nutation of the inner core. Moreover, a closer examination of

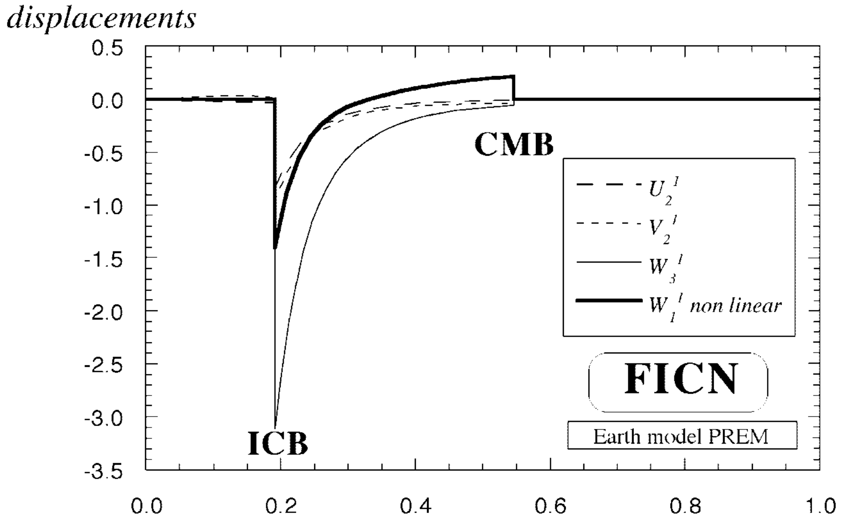

Figure 5. Same as Fig. 3 for the FICN. Near the CMB, the non-linear part of W_1^1 is clearly greater than its linear part.

Table 2. Periods (in *sidereal days*) of the FCN and FICN and corresponding relative rotation angle between the outer or inner core and the mantle. The negative (resp. positive) sign of an eigenperiod reflects the retrograde (resp. prograde) motion of the rotation axis in space. ε(ICB) and ε(CMB) are the geometrical flattenings at the ICB and CMB, respectively. $\Delta\rho_0$(ICB) is the density jump at the ICB (in g cm^{-3}).

Model	ε^{-1} (ICB)	ε^{-1} (CMB)	$\Delta\rho_0$ (ICB)	FCN			FICN		
				Period	$\tilde{\zeta}_f/\tilde{\zeta}$	$\tilde{\zeta}_s/\tilde{\zeta}$	Period	$\tilde{\zeta}_f/\tilde{\zeta}$	$\tilde{\zeta}_s/\tilde{\zeta}$
PREM	412.9	392.5	0.597	−458.6	−7.83	−4.27	473.9	11.4	−3025
PEM-c	412.2	392.4	0.565	−457.5	−7.77	−4.23	468.4	11.5	−3093
PREMM	403.1	386.3	0.368	−448.7	−7.99	−4.33	433.7	11.7	−3147
CGGM	398.9	386.2	0.000	−448.7	−7.99	−4.28	388.4	11.0	−3169

this function in the outer core and in the mantle reveals a well-marked non-linear trend near the ICB, a fact that contradicts the hypothesis of de Vries & Wahr (1991), as well as that of Mathews *et al.* (1991a) that the rotational part of the displacement is greater than the deformational part by a factor of the order of the inverse of the ellipticity. However, since the portion of the outer core where the behaviour of W_1^1 is not quasi-linear is not very large, and since the amplitude of the motion in the outer core is much smaller than in the inner core, this incorrect assumption should not influence their results for the FICN to a large extent. Numerical experimentation shows that, unlike the FCN period, which depends mainly on the dynamical ellipticity at the CMB and on the ratio between the equatorial moments of inertia of the whole Earth and the mantle, the FICN period is very sensitive to core structure. In particular, the density jump at the ICB plays an important role, as can be seen in Table 2. This table contains data and results relating to four earth models: PREM (Dziewonski & Anderson 1981), PEM-C (Dziewonski *et al.* 1975) and CGGM and PREMM (Denis *et al.* 1997). The first two models were obtained by inversion techniques by fitting a large series of data comprising seismic normal mode periods and an inertia coefficient $y=0.3308$. CGGM and PREMM are modified versions of PREM that possess an inertia coefficient $y=0.33224$. The latter leads to a surface flattening $\varepsilon(R)=1/298.3$, against 1/299.9 for PREM. Their structure differs from PREM mainly in the core. CGGM and PREMM have not been obtained by a general inversion procedure, but their eigenperiods agree well with all the observed seismic free periods.

We should add that, for the FICN, the correlation coefficient of the linear regression applied to the W_1^1 function in the outer core varies from 0.905 for CGGM to 0.989 for PEM-C (for PREM it is 0.965). This confirms the non-linear trend of W_1^1 mentioned above.

5 CONCLUSIONS

Having justified the introduction of second-order terms in ellipticity in the equations of motion of a slowly rotating, slightly elliptical earth derived by Smith (1974), we may state that the normal mode computation used in this paper is also an efficient tool for exploring the dynamics of the solid inner core. We have calculated the eigenperiods and eigenfunctions of the FCN and FICN, which are nearly diurnal wobbles in a rotating reference frame, for different realistic earth models that are assumed to be initially in hydrostatic equilibrium. Our results agree with those of de Vries & Wahr (1991) and Mathews *et al.* (1991b). Moreover, we have shown that one of the assumptions underlying their method, that is, that the deformational part of the displacement is everywhere smaller than the rotational part, may not hold in the fluid core, near the ICB, if we consider the FICN. We defer to another paper the study of the Chandler wobble of the mantle and of the Chandler wobble of the inner core, as well as the possible influence of the inertial gravity modes of the liquid outer core on these modes.

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