Geophys. J. Int. (1998) **132,** 603–642

Methods for computing internal flattening, with applications to the Earth's structure and geodynamics

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Accepted 1997 September 22. Received 1997 July 31; in original form 1996 July 1

SUMMARY

After general comments (Section 1) on using variational procedures to compute the oblateness of internal strata in the Earth and slowly rotating planets, we recall briefly some basic concepts about barotropic equilibrium figures (Section 2), and then proceed to discuss several accurate methods to derive the internal flattening. The algorithms given in Section 3 are based on the internal gravity field theory of Clairaut, Laplace and Lyapunov. They make explicit use of the concept of a level surface. The general formulation given here leads to a number of formulae which are of both theoretical and practical use in studying the Earth's structure, dynamics and rotational evolution. We provide exact solutions for the figure functions of three Earth models, and apply the formalism to yield curves for the internal flattening as a function of the spin frequency. Two more methods, which use the general deformation equations, are discussed in Section 4. The latter do not rely explicitly on the existence of level surfaces. They offer an alternative to the classical first-order internal field theory, and can actually be used to compute changes of the flattening on short timescales produced by variations in the LOD. For short durations, the Earth behaves elastically rather than hydrostatically. We discuss in some detail static deformations and Longman's static core paradox (Section 5), and demonstrate that in general no static solution exists for a realistic Earth model. In Section 6 we deal briefly with differential rotation occurring in cylindrical shells, and show why differential rotation of the inner core such as has been advocated recently is incompatible with the concept of level surfaces. In Section 7 we discuss first-order hydrostatic theory in relation to Earth structure, and show how to derive a consistent reference Earth model which is more suitable for geodynamical modelling than are modern Earth models such as 1066-A, PREM or CORE11. An important result is that a consistent application of hydrostatic theory leads to an inertia factor of about 0.332 instead of the value 0.3308 used until now. This change automatically brings 'hydrostatic' values of the flattening, the dynamic shape factor and the precessional constant into much better agreement with their observed counterparts than has been assumed hitherto. Of course, we do not imply that non-hydrostatic effects are unimportant in modelling geodynamic processes. Finally, we discuss (Sections 7–8) some implications of our way of looking at things for Earth structure and some current problems of geodynamics. We suggest very significant changes for the structure of the core, in particular a strong reduction of the density jump at the inner core boundary. The theoretical value of the free core nutation period, which may be computed by means of our hydrostatic Earth models CGGM or PREMM, is in somewhat better agreement with the observed value than that based on PREM or 1066-A, although a significant residue remains. We attribute the latter to inadequate modelling of the deformation, and hence of the

* Paper written while on sabbatical leave at: Centre de Recherches en Histoire des Sciences et des Techniques, Cité des Sciences et de l'Industrie, La Villette, Paris, France.

change in the inertia tensor, because the static deformation equations were used. We argue that non-hydrostatic effects, though present, cannot explain the large observed discrepancy of about 30 days.

Key words: Earth models, Earth structure, equilibrium figures, free core nutation, internal flattening, rotation.

1 INTRODUCTION

Abad, Pacheco & Sañudo (1995) recently proposed two variational procedures to compute the internal flattening induced in the hydrostatic structure of slowly rotating planets, and applied them to the Earth. Their paper nicely illustrates the general applicability of variational techniques, which is the main advantage of such methods, and the frequent failure to provide accurate results or to give any *a priori* hint on the accuracy attained. The latter constitutes an essential drawback of the variational approach.

Modern geodynamic theories, in particular the theory of nutation of a deformable Earth model and the theory of core modes, depend rather critically on the flattening of internal strata, especially on the flattening of the core–mantle boundary (CMB) and the inner core boundary (ICB). They require, therefore, a precise determination of the flattening of level surfaces within the Earth. In this respect, variational techniques such as those described in Abad *et al.* (1995) obviously miss the target.

However, the objective of our work is not to discuss the advantages or shortcomings of variational techniques, but rather to provide a thorough review of some concepts of the theory of figures leading to efficient computational algorithms, and apply these concepts and algorithms to the problems of Earth structure and geodynamics. More specifically, after having defined some basic concepts and notations used throughout this paper, we present briefly the integral and differential approaches to internal gravity field theory, and give the explicit figure equations for the first-, second- and third-order approximations. We then provide an alternative approach based on the small deformation equations, which, in the case of an entirely fluid model, yield the same results as the integration of Clairaut's equation. However, this second approach has the advantage of allowing one to deal with non-hydrostatic rotational deformation as well. Non-hydrostatic effects should necessarily be considered if we wish to evaluate the change of the equipotential surfaces caused by non-secular variations of Earth's centrifugal potential. We discuss the change of level surfaces and associated kinetic parameters due to variations of the Earth's rotation rate, as well as the effect of rotation on Earth structure, and show how a consistent Earth model may be derived which is more suitable for geodynamic modelling than are modern Earth models such as 1066-A, PREM or CORE11. Finally, we provide theoretical values for important kinetic parameters, in particular for the dynamical ellipticity of the core. The latter is a critical parameter in earth tide and nutation theory.

2 BASIC CONCEPTS AND NOTATIONS

Let s denote the radius of a sphere comprising the same volume as the rotationally distorted spheroidal stratum characterized by a geometrical flattening f. The parameter s, which ranges from a value of 0 at the centre to R at the outer surface ($R=6371$ km in the case of the Earth), represents, together with the colatitude θ, a set of Lyapunov variables (s, θ) associated with the hydrostatic equilibrium figure we want to determine (e.g. Denis 1989). In the following, we often use the dimensionless variable $\sigma=s/R$, which ranges from 0 to 1, instead of s itself. Hence, our Lyapunov variables are σ and θ. The variable σ, or equivalently the variable s, determines unambiguously any level surface within the rotationally distorted model.

Taking account of the symmetry properties of barotropic equilibrium figures (e.g. Denis 1989), the equation of the level surface corresponding to a particular value s may be written in Lyapunov variables,

$$r(s, \theta)=s\,[\,1+s_0(s)+s_2(s)\,P_2(\cos\theta)+s_4(s)\,P_4(\cos\theta)+\cdots\,], \tag{1}$$

if we fix the origin of the axes at the mass centre of the body. The functions $P_0, P_2, P_4, \ldots$ are Legendre polynomials of even degrees, and the corresponding figure functions $s_0, s_2, s_4, \ldots$, which must be determined, contain all the necessary information about the rotational distortion.

The equatorial radius of the spheroid corresponds to the particular value $\theta=90°$, i.e.

$$a(s)=r(s, 90°)=s\left[1+s_0(s)-\frac{1}{2}\,s_2(s)+\frac{3}{8}\,s_4(s)-\cdots\right], \tag{2}$$

whereas the polar radius is obtained for the particular value $\theta=0°$, i.e.

$$c(s)=r(s, 0°)=s[1+s_0(s)+s_2(s)+s_4(s)+\cdots]. \tag{3}$$

The geometrical flattening of any internal level surface is

$$f(s)=\frac{a(s)-c(s)}{a(s)}=-\frac{3}{2}\,s_2(s)-\frac{3}{4}\,s_2{}^2(s)-\frac{5}{8}\,s_4(s)+\cdots. \tag{4}$$

The classical Clairaut–Laplace–Lyapunov (CLL) theory of equilibrium figures, discussed in detail in the works of Zharkov & Trubitsyn (1978), Lanzano (1982) and Denis (1985, 1989), determines the figure functions from a set of integro-differential figure equations to an arbitrary order of accuracy fixed at the beginning. This can be achieved by means of a series expansion to any pre-defined order of all relevant quantities in terms of the small dimensionless quantity $m = \Omega^2 R^3 / GM$, which, for the Earth, has the value 3.449786×10^{-3} (Moritz 1980, 1990). Ω denotes the angular rotation speed and GM is the geocentric (or planetocentric) constant, thus the parameter m represents the ratio of centrifugal force to gravity force at the equator. The CLL theory works well because it establishes a hierarchy in such a way that all quantities associated with the nth figure equation ($n \geq 1$) are of the order $o(m^n)$ or smaller. In particular, the figure functions $s_2, s_4, \ldots s_{2n} \ldots$ are respectively of the orders of smallness $o(m^n)$. In fact, the figure function s_0 is *not* determined by a figure equation, but is obtained from the coupling equation

$$\sum_{n=0}^{\infty} s_{2n}(s) P_{2n}(\cos\theta_o) = 0\,, \tag{5}$$

expressing the fact that for some value $\theta = \theta_o$ of the colatitude, the distance of the level surface to the mass centre is equal to the mean radius of the stratum. Using the definition of s which readily leads to

$$\int_{-1}^{+1} \left[1 + \sum_{n=0}^{\infty} s_{2n}(s) P_{2n}(z) \right]^3 dz = 2\,, \tag{6}$$

we obtain

$$s_0(s) = -\frac{1}{5} s_2{}^2(s) - \frac{2}{105} s_2{}^3 \ldots . \tag{7}$$

The latter relation shows that s_0 is of the order of smallness $o(m^2)$. Eq. (7) should not be interpreted as a contraction of the material volume comprised by any level surface, but only as a relabelling of this surface. Indeed, a basic assumption of the CLL theory is that rotational distortion occurs in an incompressive way. The presence of a term $s_0(s)$ in eq. (1) is made necessary by this assumption.

3 SOME RESULTS OF THE INTERNAL GRAVITY THEORY

In terms of the Lyapunov variables, the effective gravity potential, which is the sum of the gravitational and centrifugal potentials, can be written

$$U(s, \theta) = -\frac{4}{3} \pi G \bar{\rho} s^2 \sum_{n=0}^{\infty} F_{2n}(s) P_{2n}(\cos\theta)\,. \tag{8}$$

The symbol $\bar{\rho}$ denotes the mean density of the Earth (or planet), i.e.

$$\bar{\rho} = \frac{3}{R^3} \int_0^R \rho(s) s^2 \, ds\,, \tag{9}$$

where $\rho(s)$ is the density of the stratum referred to by s. Throughout this paper, the word 'potential' denotes interaction energy, not work. We use, therefore, a minus sign where geodesists or astronomers would commonly use a plus sign. Eq. (8) is obtained by expanding powers such as r^k occurring in the expression of the effective gravity potential $U(r, \theta)$ in polar coordinates by means of the general binomial formula (valid for both $k > 0$ and $k < 0$):

$$r^k = s^k \left\{ 1 + \sum_{j=1}^{\infty} \frac{k(k-1)(k-2)\ldots(k-j+1)}{j!} \left[\sum_{n=0}^{\infty} s_{2n}(s) P_{2n}(\cos\theta) \right]^j \right\}, \tag{10}$$

and by converting products of Legendre polynomials, which arise in this conversion, into sums of Legendre polynomials by applying the Adams–Neumann formula (Whittaker & Watson 1969) repeatedly:

$$P_m(\cos\theta) P_n(\cos\theta) = \sum_{j=0}^{m} \frac{A_{m-j} A_j A_{n-j}}{A_{n+m-j}} \frac{2n+2m-4j+1}{2n+2m-2j+1} P_{n+m-2j}(\cos\theta)\,, \tag{11}$$

with $m \leq n$ and $A_m = (2m-1)!!/m!$.

The use of spherical harmonics in the theory of equilibrium figures raises a convergence problem, which had been acknowledged without solution for more than a century. A satisfactory solution which laid the foundation of the CLL theory on firmer grounds was first given by Trubitsyn (1972) and Trubitsyn, Vasilyev & Efimov (1976). Some details concerning this problem have been provided in the works of Zharkov & Trubitsyn (1978) and Denis (1985, 1989).

By definition, the effective gravity potential $U(s, \theta)$ is constant on any level surface specified by a fixed value of the parameter s. Thus, in the general expression (8) all the coefficients $F_{2n}(s)$ must be zero for $n \neq 0$. The case $n = 0$ provides the value of $U(s, \theta)$ on the level surface labelled s, i.e.

$$U(s) = -\frac{4}{3}\pi\bar{\rho}s^2F_0(s)\,. \tag{12}$$

Hence, the actual figure equations are given by

$$F_{2n}(s) = 0\,, \qquad n = 1, 2, 3, \dots, N, \tag{13}$$

where N is arbitrarily large. Eq. (13) in fact represents a system of N integro-differential equations.

There are basically two types of methods available for solving the system (13). The first goes back to Clairaut and Laplace, and has been used under various formulations by most authors in actual computations, generally limited to first-order (Clairaut–Laplace) or second-order (Darwin–de Sitter) approximations. It reduces the integral figure equations to a sequence of differential boundary value problems. The latter are solved by a numerical scheme. In the following, we shall refer to these methods as differential methods. A first application of this type of method to a realistic (Bullen A-type) Earth model was made by Bullard (1948), who carried out his computations to $o(m^2)$. Kopal (1960) proposed a general approach to the Clairaut–Laplace theory which at the start is quite similar to that derived in detail by Zharkov & Trubitsyn (1978) and Denis (1985, 1989), but at the end is put into a differential form. Kopal's theory, which was originally formulated to $o(m)$ and $o(m^2)$, was later extended by Lanzano (1962, 1974) and by Kopal himself (1973) to $o(m^3)$, and by Kopal & Mahanta (1974) to $o(m^4)$. However, the latter paper contains some numerical misprints which, in actual calculations, introduce errors of $o(m^3)$. Numerical calculations using the differential approach to $o(m^2)$ were performed for realistic Earth models by James & Kopal (1963) and by Nakiboğlu (1976, 1979). Similar third-order calculations were made by Lanzano & Daley (1977). Lanzano's (1982, Chapter 2) monograph contains a detailed review of Kopal's algorithmic scheme, with formulae correct to $o(m^3)$.

The second type of methods for solving the figure equations was advocated by Jeffreys (1953, 1963). It consists of solving the integral figure equations (13) directly by iteration. After extensive use of both types of methods, we believe that the integral methods are generally superior to the differential methods, at least for orders higher than second.

With the present-day terrestrial spinning rate, the flattening of strata within the Earth is smaller than 0.0034. Thus, for geodynamic purposes, it is usually sufficient to rely on first-order theory. For more precise geodetic and astronomical purposes, a second-order approximation may be desirable. The classical figure theory to $o(m^2)$ is that of Darwin (1899) and de Sitter (1924). A short derivation of this theory in a planetological context is provided by DeMarcus (1959).

For faster spinning rates, such as those encountered in the major planets and, presumably, in the Earth in a very remote past, higher-order approximations become necessary. This was essentially the reason which pushed some authors to extend the theory beyond second order. Though straightforward, the analytical derivation in a traditional manner of the explicit figure equations beyond $o(m^2)$ soon becomes rather tedious and hazardous. Therefore, no exact analytical expressions for the functions $F_{2n}(s)$ seem to be available beyond $o(m^3)$. It is thought possible, in principle, to derive higher-order figure equations using two general techniques, but none seems to have been put into operation systematically: the first technique is numerical sieving of terms in a computer; the second is handling symbolic expressions in a computer environment such as Maple, Mathematica or Reduce.

In the present investigation, however, we are essentially concerned with the derivation of efficient numerical algorithms for computing the internal flattening of a slowly rotating planet.

3.1 First-order internal gravity field theory

Clairaut's classical first-order theory (Clairaut 1743) corresponds to the case $N = 1$. All the information is contained in the figure function $s_2(s)$, which is obtained by solving the figure equation

$$F_2(s) =: -s_2(s)S_0(s) + S_2(s) + T_2(s) - \frac{m}{3} = 0\,. \tag{14}$$

To $o(m)$, the integral quantities S_0, S_2, T_2 are as follows:

$$S_0(s) = \frac{3}{\bar{\rho}s^3}\int_0^s \rho(s)s^2\,ds\,, \tag{15}$$

$$S_2(s) = \frac{3}{5\bar{\rho}s^5}\int_0^s \rho(s)\frac{d}{ds}[s^5 s_2(s)]\,ds\,, \tag{16}$$

$$T_2(s) = \frac{3}{5\bar{\rho}}\int_s^R \rho(s)\frac{d}{ds}[s_2(s)]\,ds\,. \tag{17}$$

It is easy to see that $S_0(s)$ represents the mean density of the material contained within the volume bounded by the level surface s, expressed in units of the global mean density $\bar{\rho}$. Obviously, this quantity does not depend on the internal flattening and is a quantity of $o(1)$. The functions $S_2(s)$ and $T_2(s)$, on the other hand, depend on the internal flattening, and are quantities of $o(m)$. They are related to the changes of the principal moments of inertia caused by the rotational distortion.

3.1.1 First-order theory: integral approach

In the integral approach, the figure function $s_2(s)$ is easily obtained if we write eq. (14) in the following form ready for iteration:

$$s_2^{(k+1)}(s)=G_2[m,\, s_2^{(k)}(s)], \quad (k=0,\, 1,\, 2,\, \ldots)\,. \tag{18}$$

We start the iterative procedure from scratch, i.e. $s_2^{(0)}(s)=0$, and we stop the integration scheme after the Kth integration such that $|s_2^{(K)}(s)-s_2^{(K-1)}(s)|\le\epsilon$ for every value of s in the interval $[0, R]$, ϵ being a conveniently chosen tolerance, e.g. $\epsilon=10^{-6}$.

3.1.2 First-order theory: differential approach

According to eq. (4), we obtain $s_2(s)=-2f(s)/3$ if we neglect quantities of $o(m^2)$ and smaller. Replacing s_2 by f in eq. (14) yields Clairaut's equation in integral form, viz.

$$fS_0+\frac{3}{2}\,(S_2+T_2)-\frac{m}{2}=0\,. \tag{19}$$

In terms of the dimensionless variables $\sigma=s/R$ and $\delta=\rho/\bar{\rho}$, the quantities S_0, S_2 and T_2 become respectively

$$S_0(\sigma)=\frac{3}{\sigma^3}\int_0^\sigma \delta(\sigma)\sigma^2\,d\sigma\,, \tag{20a}$$

$$S_2(\sigma)=-\frac{2}{5\sigma^5}\int_0^\sigma \delta(\sigma)\,\frac{d(f\sigma^5)}{d\sigma}\,d\sigma\,, \tag{20b}$$

$$T_2(\sigma)=-\frac{2}{5}\int_\sigma^1 \delta(\sigma)\,\frac{df}{d\sigma}\,d\sigma\,. \tag{20c}$$

Clairaut's equation thus appears as a linear integro-differential equation for the flattening of internal strata. It may be solved iteratively by the general procedure indicated above for the figure function s_2. However, the classical approach is to differentiate (19) with respect to σ and use the relations (20) to eliminate m and T_2. In this way we obtain the intermediate result

$$-\frac{5}{2}\,S_2=\left(f-\frac{\sigma}{3}\frac{df}{d\sigma}\right)S_0\,. \tag{21}$$

Multiplying the latter by σ^5 and differentiating the result again with respect to σ yields *Clairaut's differential equation*,

$$\frac{d^2f}{d\sigma^2}+\frac{6\delta}{\sigma S_0}\frac{df}{d\sigma}+\frac{6}{\sigma^2}\left(\frac{\delta}{S_0}-1\right)f=0\,. \tag{22}$$

This equation can be integrated numerically subject to the conditions that f be finite at the centre ($\sigma=0$) and fulfil at the surface ($\sigma=1$) the constraint

$$\left[\frac{df}{d\sigma}+2f\right]_{\sigma=1}=\frac{5}{2}\,m\,. \tag{23}$$

We obtain the boundary condition (23) if we multiply (19) by σ^5, differentiate the result with respect to σ, then put $\sigma=1$ and note that $S_0(1)=1$.

Eq. (22) with the boundary condition (23) is most easily solved nowadays on a digital computer by putting $y_1=f$ and $y_2=df/d\sigma$. The ensuing second-order differential system is

$$\frac{d}{d\sigma}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ \dfrac{6}{\sigma^2}\left(1-\dfrac{\delta}{S_0}\right) & -\dfrac{6\delta}{\sigma S_0} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tag{24}$$

with the boundary conditions (i) $|y_1(0)|<\infty$ and (ii) $y_2(1)+2y_1(1)=5m/2$. We notice that eq. (22), or alternatively the system (24), has the origin as a regular singular point in the Fuchsian sense (e.g. Båth & Berkhout 1984, p. 70; Coddington & Levinson 1955).

Let us suppose, in accordance with PREM and the two recently suggested models CGGM and PREMM, that the density in the vicinity of the origin is given as

$$\rho(\sigma)=\rho_c(1+\alpha\sigma^2)\,, \tag{25}$$

then the average density $D(\sigma)=\bar{\rho}S_0(\sigma)$ within the sphere of radius σ is

$$D(\sigma)=\rho_c\left(1+\frac{3}{5}\,\alpha\sigma^2\right). \tag{26}$$

For PREM (Dziewonski & Anderson 1981) the value of the central density ρ_c is 13088.5 kg m^{-3}, and the dimensionless constant α is -0.675257. For CGGM and PREMM (Denis *et al.* 1997) we have $\rho_c=12279.6$ kg m^{-3} and $\rho_c=12588.5$ kg m^{-3} respectively, and $\alpha=-0.665307$ and $\alpha=-0.702077$ respectively . It is noteworthy that no linear term in σ can occur in a Taylor expansion near the origin. This results from the fact that the density gradient must vanish at the centre. Indeed, if we recall the structure equation

$$\frac{d\rho}{ds}=-\frac{\eta\rho g}{\phi}, \tag{27}$$

where $\eta\neq 0$ is the Bullen factor and $\phi\neq 0$ is the seismological factor, we see that $[d\rho/d\sigma]_{\sigma=0}=0$ because gravity g vanishes at the origin.

Fuchs' theorem entails that a solution of (22) exists in the neighbourhood of the origin in the particular form

$$f(\sigma)=\sum_{i=0}^{\infty} c_i\sigma^{\gamma+i}, \tag{28}$$

with $c_0\neq 0$. Moreover, we may write successively

$$\frac{df}{d\sigma}=\sum_{i=0}^{\infty} c_i(\gamma+i)\sigma^{\gamma+i-1}, \tag{29}$$

$$\frac{d^2f}{d\sigma^2}=\sum_{i=0}^{\infty} c_i(\gamma+i)(\gamma+i-1)\sigma^{\gamma+i-2}. \tag{30}$$

To determine the coefficients c_i, $i=0, 1, 2, \ldots$, we may apply the Frobenius procedure (Coddington & Levinson 1955) to eq. (22) written in the form

$$\left(1+\frac{3}{5}\alpha\sigma^2\right)\frac{d^2f}{d\sigma^2}+(1+\alpha\sigma^2)\frac{6}{\sigma}\frac{df}{d\sigma}+\frac{12}{5}\alpha f=0. \tag{31}$$

The latter is valid for $0\leq\sigma<\sigma_c$, where $\sigma_c=\sqrt{-5/(3\alpha)}$. Thus the solution obtained is valid in the whole inner core of PREM, for which $\sigma_c=1.571$.

Putting (28–30) into eq. (31) yields

$$\gamma(\gamma+5)c_0\sigma^{\gamma-2}+(\gamma+1)(\gamma+6)c_1\sigma^{\gamma-1}+\sum_{i=0}^{\infty}[\varphi(i)c_{i+2}+\psi(i)c_i]\sigma^{\gamma+i}=0, \tag{32}$$

with $\varphi(\gamma, i)=(\gamma+i+2)(\gamma+i+7)$ and $\psi(\gamma, i)=(3/5)\alpha[(\gamma+i)(\gamma+i+9)+4]$. Because eq. (32) has to be fulfilled for any value of $\sigma<\sigma_c$, the coefficients of the individual powers of σ must vanish separately. Moreover, because $c_0\neq 0$, we have $\gamma=0$ or $\gamma=-5$. The boundedness of f at the origin implies that the most general physical solution of eq. (31) corresponds to the root $\gamma=0$ of the indicial equation $\gamma(\gamma+5)=0$. This root implies that $c_1=0$ for the equation $(\gamma+1)(\gamma+6)c_1=0$ to be fulfilled. The coefficients c_i for $i>1$ can be obtained recursively by means of the formula

$$c_{i+2}=-\frac{\psi(0, i)}{\varphi(0, i)}c_i=-\frac{3\alpha}{5}\frac{i^2+9i+4}{(i+2)(i+7)}c_i, \qquad i=0, 1, 2, 3, \ldots. \tag{33}$$

Because this recursion formula involves at each step i only the coefficients c_{i+2} and c_i, it is at once obvious that only coefficients with an even index can be non-zero. Putting $c_0=1$ and denoting the flattening at the centre as f_c, the flattening f within the inner core and its derivative $df/d\sigma$ are provided by

$$f(\sigma)=f_c\sum_{k=0}^{\infty} c_{2k}\sigma^{2k}, \tag{34}$$

$$\frac{df(\sigma)}{d\sigma}=f_c\sum_{k=0}^{\infty} 2k\, c_{2k}\sigma^{2k-1}. \tag{34$'$}$$

It is a matter of a few lines of code to evaluate the expansion coefficients c_{2k} numerically by means of the recursion formula (33) for any given value of α. Moreover, it is possible to decompose each individual coefficient c_{2k} into a factor α_k depending on the particular model considered, and on a factor γ_k which is the same for all models, i.e.

$$c_{2k}=\alpha_k\gamma_k \quad (k=0, 1, 2, 3, \ldots), \tag{35}$$

where

$$\alpha_k=\left(-\frac{3\alpha}{5}\right)^k, \qquad \beta_k=\frac{4k^2+18k+4}{4k^2+18k+14}, \quad \gamma_0=1, \quad \gamma_{k+1}=\beta_k\gamma_k. \tag{36}$$

Stated differently, eq. (35) reads

$$c_0=1\,,\qquad c_{2k}=\left(-\frac{3\alpha}{5}\right)^k\prod_{j=1}^{k}\beta_j\,,\qquad k=1,\,2,\,3,\,4,\,\ldots\,. \tag{35'}$$

Because $\alpha<0$ all the coefficients c_{2k} in the expansion (34) are non-negative.

In particular, we notice that for a model with constant density from the centre to the surface, $\alpha=0$. Thus, all coefficients c_{2k} except c_0 vanish, implying that the flattening is constant everywhere and is provided by its value at the surface. The latter is easily obtained as $f=5m/4$ from the boundary condition (23), where we put $df/d\sigma=0$. This is a classical result. For a Roche-type model, that is a planetary model possessing a density structure given by eq. (25) from the centre to the surface, the series expansion (34) determines uniquely the flattening of each level surface. Thus, a simple density law accounting for the Earth's mean density ($\bar{\rho}=5515$ kg m^{-3}) and inertia coefficient ($\bar{y}=0.331$) is $\rho(\sigma)=10510(1-0.792\sigma^2)$. Table 1 provides the relevant expansion coefficients c_{2k} for this Roche model, as well as for the realistic Earth models PREM of Dziewonski & Anderson (1981) and CGGM and PREMM of Denis *et al.* (1997).

For PREM and PREMM, the expansion (34) holds up to the ICB at $\sigma_{\rm ICB}=0.191728$, for CGGM it holds up to the CMB at $\sigma_{\rm CMB}=0.546225$, and for the Roche model it holds up to the outer surface at $\sigma_1=1.0$. In agreement with Clairaut's approximation and the present-day slow spinning rate of the Earth, we should retain only the coefficients which secure a global accuracy of four significant digits. For PREM and PREMM, only the coefficients c_0, c_2, c_4 are needed to represent f with four significant digits throughout the inner core, and only the coefficients c_2, c_4, c_6 are needed to represent $df/d\sigma$. For CGGM, the coefficients c_0, c_2, c_4, c_6, c_8 yield the internal flattening f with four significant digits from the centre to the CMB, and the coefficients c_2, c_4, c_6, c_8, c_{10}, c_{12} are necessary for $df/d\sigma$. For the Roche model, the coefficients $c_0, c_2, \ldots, c_{18}$ represent f throughout the body with four significant figures, and the coefficients $c_2, c_4, \ldots, c_{26}$ achieve the same for $df/d\sigma$.

The constant f_c must be adapted in such a way as to fulfil the boundary condition at the surface, eq. (23). This can be achieved as follows.

(1) Start integration of the differential system (24) at $\sigma=\sigma_{\rm ICB}$ by means of a numerical integration code such as the Runge–Kutta or the Bulirsch–Stoer schemes (Press *et al.* 1992) using the starting values $y_1=y_1(\sigma_{\rm ICB})$ and $y_2=y_2(\sigma_{\rm ICB})$. In the case of PREM, the respective numerical values are: $\sigma_{\rm ICB}=0.19173$, $y_1(\sigma_{\rm ICB})=1.004292694046$ and $y_2(\sigma_{\rm ICB})=0.045172626651$.

(2) Integrate up to the surface to obtain values $y_1(1)$, $y_2(1)$ at $\sigma=1$.

(3) Determine the central flattening by means of the relation

$$f_c=\frac{2.5m}{[df/d\sigma+2f]_1}\,.$$

We readily obtain, in the case of PREM, $f_c=0.00240994=1/414.95$.

Table 1. The universal coefficients β_k and γ_k for $k=0, 1, 2, \ldots,$ 20 and the corresponding expansion coefficients c_{2k} for the Earth models PREM (Dziewonski & Anderson 1981), CGGM and PREMM (Denis *et al.* 1997) and the classical Roche model.

k	β_k	γ_k	c_{2k} (PREM)	c_{2k} (CGGM)	c_{2k} (PREMM)	c_{2k} (Roche)
0	0.28571429	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
1	0.72222222	0.28571429	0.11575834	0.11405263	0.12035606	0.13577143
2	0.84848485	0.20634921	0.03387221	0.03288134	0.03661633	0.04659675
3	0.90384615	0.17508418	0.01164415	0.01113697	0.01308745	0.01878781
4	0.93333333	0.15824916	0.00426405	0.00401823	0.00498294	0.00806951
5	0.95098039	0.14769921	0.00161243	0.00149708	0.00195911	0.00357899
6	0.96240602	0.14045906	0.00062126	0.00056832	0.00078481	0.00161737
7	0.97023810	0.13517864	0.00024224	0.00021833	0.00031817	0.00073968
8	0.97584541	0.13115547	0.00009522	0.00008456	0.00013004	0.00034103
9	0.98000000	0.12798746	0.00003765	0.00003294	0.00005346	0.00015814
10	0.98316498	0.12542771	0.00001495	0.00001289	0.00002207	0.00007365
11	0.98563218	0.12331613	0.00000595	0.00000506	0.00000914	0.00003441
12	0.98759305	0.12154435	0.00000238	0.00000199	0.00000379	0.00001612
13	0.98917749	0.12003636	0.00000095	0.00000078	0.00000158	0.00000756
14	0.99047619	0.11873726	0.00000038	0.00000031	0.00000066	0.00000356
15	0.99155405	0.11760643	0.00000015	0.00000012	0.00000027	0.00000167
16	0.99245852	0.11661313	0.00000006	0.00000005	0.00000011	0.00000079
17	0.99322493	0.11573370	0.00000002	0.00000002	0.00000005	0.00000037
18	0.99388005	0.11494959	0.00000001	0.00000001	0.00000002	0.00000018
19	0.99444444	0.11424611	0.00000000	0.00000000	0.00000001	0.00000008
20	0.99493414	0.11361141	0.00000000	0.00000000	0.00000000	0.00000004

(4) Multiply all values computed from series expansion or numerical integration by f_c to obtain the actual distribution of f as a function of σ. In particular, the PREM value for the surface flattening is $0.00333439 = 1/299.90$.

3.1.3 First-order theory: Radau's approximation

Nowadays, with the general availability of digital computers, the computation of the internal flattening using the methods outlined in the previous sections is rather trivial. However, before modern means of computation became widely available, much effort was spent on finding ways to make the integration of Clairaut's equation feasible. In this respect, the work of Radau (1885) deserves special notice, not only because it allows the integration of eq. (22) with minimal numerical effort, but also because it provides several important formulae for the theory of the Earth's hydrostatic structure. For this reason, Radau's theory has become a rather standard geophysical approach to assessing the hydrostatic shape of the Earth (see for instance Lambert & Darling 1951; Jeffreys 1970, pp. 187–189).

Differentiating eq. (20a) with respect to σ yields the relation between the density δ on the level surface of mean radius σ and the mean density S_0 of the material in the volume contained within this level surface, namely

$$\delta(\sigma) = S_0(\sigma) + \frac{\sigma}{3}\frac{dS_0}{d\sigma}. \tag{37}$$

Multiply eq. (22) by $\sigma^2 S_0$, replace δ with its expression (37) in the result, and rearrange the terms with respect to S_0 and $dS_0/d\sigma$. The resulting equation is

$$\left(\sigma^2\frac{d^2f}{d\sigma^2} + 6\sigma\frac{df}{d\sigma}\right)S_0 + 2\left(\sigma^2\frac{df}{d\sigma} + \sigma f\right)\frac{dS_0}{d\sigma} = 0. \tag{38}$$

Following Radau (1885), let us introduce the dimensionless variable

$$\eta = \frac{\sigma}{f}\frac{df}{d\sigma}. \tag{39}$$

A number of people concerned with the theory of the equilibrium shape and structure of the Earth and slowly rotating planets have tried to provide a simple geometrical interpretation of Radau's parameter η, apparently without success. With this new variable, eq. (38) becomes

$$\left(\sigma\frac{d\eta}{d\sigma} + \eta^2 + 5\eta\right)S_0 + 2\sigma(\eta+1)\frac{dS_0}{d\sigma} = 0. \tag{40}$$

This can be put into the form

$$\frac{d}{d\sigma}(\sqrt{\eta+1}\,S_0) + \frac{5\eta+\eta^2}{2\sigma\sqrt{\eta+1}}\,S_0 = 0, \tag{41}$$

as is easily checked by expanding the first term in (41). Now, if we differentiate the expression $\sigma^5\sqrt{\eta+1}\,S_0$, treating $\sqrt{\eta+1}\,S_0$ as an inseparable block, we obtain, by making use of (41),

$$\frac{d}{d\sigma}(\sigma^5\sqrt{\eta+1}\,S_0) = 5\sigma^4 F(\eta) S_0, \quad \text{where } F(\eta) = \frac{1+\frac{1}{2}\eta - \frac{1}{10}\eta^2}{\sqrt{\eta+1}}. \tag{42}$$

This first-order differential equation (42) may be called the *Clairaut–Radau equation*. It merely represents the Clairaut equation (22) written in terms of the Radau parameter η. The usefulness of the Radau transformation stems from the fact that the function $F(\eta)$ is very near unity for all values of η likely to occur in terrestrial planets. Fig. 1 shows $F(\eta)$ for $0 \le \eta \le 1$. The reason for $F(\eta)$ staying close to unity in this interval is that in its Taylor–Maclaurin expansion, namely

$$F(\eta) = 1 + \frac{1}{40}\eta^2 - \frac{3}{40}\eta^3 + \cdots,$$

the linear term cancels and the higher-order terms compensate to some extent. If we put $F(\eta)=1$, eq. (42) can be integrated immediately to

$$\sigma^5\sqrt{\eta(\sigma)+1}\,S_0(\sigma) = 5\int_0^\sigma \sigma^4 S_0(\sigma)\,d\sigma. \tag{43}$$

$S_0(\sigma)$ being a known function, eq. (43) yields η as a function of σ.

Once we have actually determined $\eta(\sigma)$, we obtain the flattening of any internal level surface from the definition (39) of η. Thus

$$f(\sigma) = f(1)\exp\left[-\int_\sigma^1 \frac{\eta(\sigma)}{\sigma}\,d\sigma\right]. \tag{44}$$

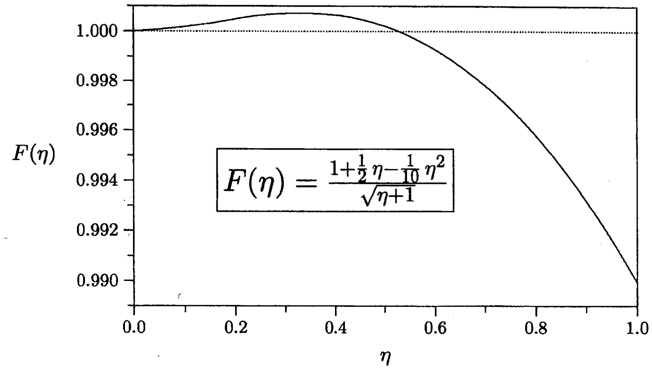

Figure 1. Behaviour of the function $F(\eta)=[1+(1/2)\eta-(1/10)\eta^2]/(\sqrt{\eta+1})$ in the interval $0\leq\eta\leq 1$. For the Earth, the maximal value of η, attained at the outer surface, is less than 0.6. In this case, $F(\eta)$ differs from 1 at most by 8×10^{-4}.

The surface flattening $f(1)$ is provided by the boundary condition (23) which, together with (39), yields

$$f(1)=\frac{5m}{2[\eta(1)+2]}\,. \tag{45}$$

Eqs. (44) and (45) are *exact* equations of the first-order theory. They are approximate formulae in the sense of Radau's approximation only if η is computed with the assumption $F(\eta)=1$. In the latter case, we may write

$$\eta(\sigma)=\left[\frac{5}{2}-\frac{15\,y(\sigma)}{4}\right]^2-1\,, \tag{46}$$

where $y(\sigma)$ is the inertia coefficient at level σ, representing the zeroth-order mass distribution factor,

$$y(\sigma)=\frac{2}{\sigma^5 S_0(\sigma)}\int_0^\sigma \delta(\sigma)\sigma^4\,d\sigma\,. \tag{47}$$

Indeed, expressed in the dimensional (physical) variables s, $\rho(s)$ and $D(s)$ instead of the non-dimensional variables σ, $\delta(\sigma)$ and $S_0(\sigma)$, respectively, eq. (47) reads

$$y(s)=\frac{\frac{8\pi}{3}\int_0^s \rho(s)s^4\,ds}{M(s)s^2}\,, \tag{47$'$}$$

if we denote by $M(s)=(4/3)\pi s^3 D(s)$ the mass of the material volume bounded by the level surface labelled s. The numerator of the right-hand side of eq. (47′) is obviously the inertia moment of a spherical body about an axis passing through the centre. Eq. (46) is readily established if in the integral of the right-hand side of (43) we substitute S_0 by $\delta-(\sigma/3)(dS_0/d\sigma)$ obtained from (37), giving

$$\sigma^5 S_0\sqrt{\eta+1}=5\int_0^\sigma \sigma^4\delta\,d\sigma-\frac{5}{3}\int_0^\sigma \sigma^5 dS_0\,,$$

integrate by parts the expression $\int_0^\sigma \sigma^5 dS_0$, which yields $\sigma^5 S_0-5\int_0^\sigma \sigma^4 S_0 d\sigma$, use again (43) to replace $5\int_0^\sigma \sigma^4 S_0 d\sigma$ with $\sigma^5 S_0\sqrt{\eta+1}$, make use of the definition (47), and finally regroup all the terms. Eq. (46), based on Radau's approximation, is particularly useful for evaluating $\eta(\sigma)$ and hence $f(\sigma)$ because simple algorithms can be provided to estimate $S_0(\sigma)$ and $\int_0^\sigma \delta(\sigma)\sigma^4 d\sigma$, and thus $y(\sigma)$ (Denis & İbrahim 1980, 1981).

An *exact* solution of the Clairaut equation can be obtained if we correct for the fact that $F(\eta)$ is not strictly 1 by means of a Newton-type iteration procedure. This may be done by writing

$$\eta^{(k)}(\sigma)=\frac{25}{\sigma^{10}{S_0}^2(\sigma)}\left\{\int_0^\sigma \sigma^4 S_0(\sigma)F[\eta^{(k-1)}(\sigma)]\,d\sigma\right\}^2-1 \qquad (k=1,\,2,\,3,\,\ldots)\,.$$

If we start the iteration scheme with $F[\eta^{(0)}(\sigma)]=1$, where $\eta^{(0)}(\sigma)$ is computed from eq. (46), convergence is extremely fast. Denis & İbrahim (1981) suggest an efficient numerical scheme to evaluate the integral occurring on the right-hand side of the latter iteration formula. We have used intensively all three methods discussed so far, namely

(1) the integral method based on solving iteratively the figure equation (18);

(2) the differential method based on solving Clairaut's equation (22) by straightforward numerical integration of the differential system (24);

(3) the immediate integration method suggested by Radau's approximation.

None of the three seems to have a definite advantage over the others. However, because of the general availability of 'canned' integrators, straightforward numerical integration of the system (24) is often quite appealing.

3.2 Second-order internal gravity field theory

Darwin's classical second-order theory (Darwin 1899) corresponds to the case $N=2$ in the general figure theory embodied in the system of figure equations (13). All the information is contained in the figure functions $s_2(\sigma)$ and $s_4(\sigma)$, which are obtained by solving the system of figure equations

$$F_2(\sigma)=:\left(-s_2+\frac{2}{7}s_2{}^2\right)S_0+\left(1-\frac{6}{7}s_2\right)S_2+\left(1+\frac{4}{7}s_2\right)T_2-\left(1-\frac{10}{7}s_2\right)\frac{m}{3}=0\,, \tag{48a}$$

$$F_4(\sigma)=:\left(-s_4+\frac{18}{35}s_2{}^2\right)S_0-\frac{54}{35}s_2S_2+\frac{36}{35}s_2T_2+S_4+T_4-\frac{12}{35}ms_2=0\,. \tag{48b}$$

To o(m^2) the integral quantities S_2, T_2, S_4, T_4 are defined as follows:

$$S_2(\sigma)=\frac{3}{5\sigma^5}\int_0^\sigma \delta\frac{d}{d\sigma}\left[\sigma^5\left(s_2+\frac{4}{7}s_2{}^2\right)\right]d\sigma\,, \tag{49}$$

$$T_2(\sigma)=\frac{3}{5}\int_\sigma^1 \delta\frac{d}{d\sigma}\left(s_2-\frac{1}{7}s_2{}^2\right)d\sigma\,, \tag{50}$$

$$S_4(\sigma)=\frac{1}{3\sigma^7}\int_0^\sigma \delta\frac{d}{d\sigma}\left[\sigma^7\left(s_4+\frac{54}{35}s_2{}^2\right)\right]d\sigma\,, \tag{51}$$

$$T_4(\sigma)=\frac{\sigma^2}{3}\int_\sigma^1 \delta\frac{d}{d\sigma}\left[\sigma^{-2}\left(s_4-\frac{27}{35}s_2{}^2\right)\right]d\sigma\,. \tag{52}$$

It is easy to see that the functions S_2 and T_2 are of o(m) and the functions S_4 and T_4 are of o(m^2). Whereas S_2 and T_2 are associated with rotationally induced changes of the inertia tensor, S_4 and T_4 are related to higher-order changes in the mass distribution caused by the rotational distortion.

3.2.1 *Second-order theory: integral approach*

In the integral approach, the figure functions $s_2(\sigma)$ and $s_4(\sigma)$ are obtained iteratively by writing the system of figure equations (48) under the following form ($k=0, 1, 2, \ldots$):

$$s_2^{(k+1)}(\sigma)=G_2[m, s_2^{(k)}(\sigma), s_4^{(k)}(\sigma)]\,, \tag{53a}$$

$$s_4^{(k+1)}(\sigma)=G_4[m, s_2^{(k+1)}(\sigma), s_4^{(k)}(\sigma)]\,. \tag{53b}$$

The iteration may either be started from scratch, i.e. $s_2^{(0)}(\sigma)=s_4^{(0)}(\sigma)=0$, or it may be started with $s_2^{(0)}(\sigma)$ provided by the first-order solution. In general, the overhead coding involved in the second alternative does not justify the slightly faster convergence of the iteration procedure.

From the coupling equation (5) we notice that s_0 is at least of order o(m). Let us expand the integrand of (6) to o(m^2) and remember that $\int_{-1}^1 P_k(z)\,dz=0$ for $k\geq 1$. We then easily obtain

$$s_0(\sigma)=-\frac{1}{5}s_2{}^2\,, \tag{54}$$

showing that s_0 is actually a small quantity of order o(m^2).

3.2.2 *Second-order theory: Darwin's approach*

The equation of a spheroid in the restricted sense, that is of an ellipsoid of revolution with equatorial radius a and flattening f, is

$$r(\theta)=\frac{a}{\sqrt{(1-f)^{-2}\cos^2\theta+\sin^2\theta}}\,. \tag{55}$$

Expanding the latter relation into powers of f, we find to $o(m^2)$ that

$$r(\theta)=a\cdot\left[1-f\cos^2\theta-\frac{3}{8}f^2\sin^2 2\theta\right]. \tag{56}$$

We rewrite the latter in terms of the Legendre polynomials $P_0=1$, P_2, P_4 by making use of the general formula

$$\cos^n\theta=\sum_{k=0}^{[\frac{n}{2}]}\frac{(2n-4k+1)n!}{(2n-2k+1)!!(2k)!!}P_{n-2k}(\cos\theta), \tag{57}$$

where the symbol $[\frac{n}{2}]$ denotes $n/2$ if n is even, and $(n-1)/2$ if n is odd. A proof of this formula can be found in the book of MacMillan (1930, pp. 352–354). Unfortunately, there is a misprint in the final formula, which we have corrected here. In particular, we obtain

$$\cos^2\theta=\frac{1}{3}+\frac{2}{3}P_2(\cos\theta)\quad\text{and}\quad\cos^4\theta=\frac{1}{5}+\frac{4}{7}P_2(\cos\theta)+\frac{8}{35}P_4(\cos\theta). \tag{58}$$

Thus, in terms of Legendre polynomials, the equation of an ellipsoid with axial symmetry to $o(m^2)$ is

$$r(\theta)=a\cdot\left[1-\frac{1}{3}f-\frac{1}{5}f^2-\frac{2}{3}f\left(1+\frac{1}{7}f\right)P_2(\cos\theta)+\frac{12}{35}f^2P_4(\cos\theta)\right]. \tag{56$'$}$$

Taking account of (2) and (4), we obtain for a level surface the following expression, valid to $o(m^2)$:

$$s=\frac{a}{1+s_0-\frac{1}{2}s_2+\frac{3}{8}s_4}=a\cdot\left(1-s_0+\frac{1}{2}s_2-\frac{3}{8}s_4+\frac{1}{4}{s_2}^2\right). \tag{59}$$

Thus, eq. (1) of a level surface may also be written to order $o(m^2)$ as follows:

$$r(\theta)=a\cdot\left[1-\frac{1}{3}f-\frac{7}{12}s_4-\frac{2}{3}\left(f+\frac{5}{8}s_4\right)P_2(\cos\theta)+s_4P_4(\cos\theta)\right]. \tag{60}$$

Unless s_4 takes on the particular value $12f^2/35$, which it does only in the case of a uniform body, we may conclude that the relations (56′) and (60) differ at $o(m^2)$, but not at $o(m)$. What this means is that in the Clairaut approximation, the earth's figure is an ellipsoid of revolution, but in the Darwin approximation it differs from such a spheroid. This is essentially the content of Hamy's (1889) theorem, and indicates that much of the beautiful work on ellipsoidal figures of equilibrium (see e.g. Chandrasekhar 1969) is of little use in problems concerned with actual cosmic bodies.

Darwin (1899) sought the equation of a level surface by introducing a second-order correction κ into eq. (56) as follows:

$$r(\theta)=a\cdot\left[1-f\cos^2\theta-\left(\frac{3}{8}f^2+\kappa\right)\sin^2 2\theta\right]. \tag{61}$$

The function $\kappa(\sigma)$ expresses the departure of the actual level surfaces from exact ellipsoids to $o(m^2)$. Because of a rather well-chosen first approximation, this correction function is very small. Comparing (61) expanded in Legendre polynomials with eq. (60), we find that

$$s_0=-\frac{4}{45}f^2,\qquad s_2=-\frac{2}{3}f-\frac{23}{63}f^2-\frac{8}{21}\kappa,\qquad s_4=\frac{12}{35}f^2+\frac{32}{35}\kappa, \tag{62}$$

or, reversely (noting that $f^2=9{s_2}^2/4$),

$$\kappa=\frac{35}{32}s_4-\frac{27}{32}{s_2}^2. \tag{63}$$

The figure equation $F_4(\sigma)=0$ (*cf.* eq. 48b) may easily be brought into the form

$$(-f^2-8\kappa)S_0+9fS_2-6fT_2+\frac{35}{4}(S_4+T_4)+2mf=0. \tag{64}$$

We replace in this equation the last term, $2mf$, by making use of Clairaut's equation in integral form (19). This is possible because the term contains the factor m, and therefore f needs to be given only to $o(m)$. In this way, we obtain Darwin's equation in integral form, in which all terms are of second-order smallness:

$$(3f^2-8\kappa)S_0+15fS_2+\frac{35}{4}(S_4+T_4)=0. \tag{65}$$

Here S_4 and T_4 must be evaluated to o(m^2), S_2 must be evaluated to o(m), and S_0 is of course of o(1) as given by eq. (20a). Therefore, we put

$$S_2(\sigma) = -\frac{2}{5\sigma^5}\int_0^\sigma \delta(\sigma)\,\frac{d f\sigma^5}{d\sigma}\,d\sigma\,, \tag{66a}$$

$$S_4(\sigma) = \frac{12}{35\sigma^7}\int_0^\sigma \delta(\sigma)\,\frac{d\left(f^2+\frac{8}{9}\,\kappa\right)\sigma^7}{d\sigma}\,d\sigma\,, \tag{66b}$$

$$T_4(\sigma) = \frac{32\sigma^2}{105}\int_\sigma^1 \delta(\sigma)\,\frac{d\kappa\sigma^{-2}}{d\sigma}\,d\sigma\,. \tag{66c}$$

As above, we notice that in eq. (65) the flattening f needs only to be evaluated to o(m), which may be achieved by solving Clairaut's equation. Thus, it becomes possible to solve Darwin's equation (64) in integral form. The solution yields the departure $\kappa(\sigma)$ of the internal level surfaces from rotational ellipsoids, correct to o(m^2).

It is obvious that eq. (65) can be put into differential form. To achieve this, we differentiate the quantities $\sigma^5 S_2$, $\sigma^7 S_4$, $\sigma^{-2} T_4$ and eliminate $df/d\sigma$, $d^2f/d\sigma^2$ by means of Clairaut's equation. Introducing Radau's parameter as defined in (39), we obtain Darwin's second-order differential equation for $\kappa(\sigma)$, which is equivalent to the integral equation (64):

$$\sigma^2\,\frac{d^2\kappa}{d\sigma^2} + \sigma\,\frac{6\delta}{S_0}\,\frac{d\kappa}{d\sigma} - \left(20 - \frac{6\delta}{S_0}\right)\kappa = \left[3 - \frac{3\delta}{S_0} + \left(1 - \frac{9\delta}{2S_0}\right)\eta - \left(\frac{1}{4} - \frac{9\delta}{4S_0}\right)\eta^2\right] f^2\,. \tag{67}$$

A first boundary condition is obtained from eqs (22) and (65), namely

$$\left[\frac{d\kappa}{d\sigma} + 4\kappa + \frac{5m}{4}\,f\right]_{\sigma=1} = \frac{25}{16}\,m^2\,. \tag{68}$$

A second boundary condition is provided by the fact that κ must be bounded at the centre. The Frobenius method applied to eq. (67) shows that the characteristic power series of $\kappa(\sigma)$ near the origin starts with a term in σ^2. Thus, $\kappa(0)=0$.

3.2.3 *Second-order theory: de Sitter's approach*

Having computed the departure $\kappa(\sigma)$ of an actual level surface from an exact ellipsoid to o(m^2), we are in a position to determine $f(\sigma)$ to o(m^2) as well. For this purpose, we retain all terms up to o(m^2) in eq. (48a) and substitute therein the function s_2 (*cf.* eq. 62b). We thus obtain the Darwin–de Sitter equation in integral form, i.e.

$$\left(f + \frac{31}{42}f^2 + \frac{4}{7}\,\kappa\right)S_0 + \frac{3}{2}\left(1 + \frac{4}{7}f\right)S_2 + \frac{3}{2}\left(1 - \frac{8}{21}f\right)T_2 - \frac{m}{2}\left(1 + \frac{20}{21}f\right) = 0\,. \tag{69}$$

In this case, the quantities S_2 and T_2 must be evaluated to o(m^2) from the relations

$$S_2(\sigma) = -\frac{2}{5\sigma^5}\int_0^\sigma \delta(\sigma)\,\frac{d\left[\left(f + \frac{1}{6}f^2 + \frac{4}{7}\,\kappa\right)\sigma^5\right]}{d\sigma}\,d\sigma\,, \tag{70a}$$

$$T_2(\sigma) = -\frac{2}{5}\int_0^\sigma \delta(\sigma)\,\frac{d\left(f + \frac{9}{14}f^2 + \frac{4}{7}\,\kappa\right)}{d\sigma}\,d\sigma\,. \tag{70b}$$

S_0 should in principle be evaluated to o(m), implying that the density $\delta(\sigma)$ should first be determined to o(m) by solving the hydrostatic equilibrium equation. We suppose here that no density changes occur from the rotational distortion. This means that we assume S_0 is always given by eq. (20a).

The Darwin–de Sitter integral equation (69) can be transformed by putting

$$f^* = f - \frac{5}{42}f^2 + \frac{4}{7}\,\kappa\,, \tag{71}$$

and generalizing the definition (39) of the Radau variable to become

$$\eta^* = \frac{\sigma}{f^*}\,\frac{df^*}{d\sigma}\,. \tag{72}$$

In this way, we obtain de Sitter's equation, namely (de Sitter 1924, eq. 16)

$$\frac{1}{5}\,\sigma^5 S_0(\sigma)\sqrt{1+\eta^*(\sigma)} = \int_0^\sigma S_0(\sigma)\sigma^4 F^*[\eta^*(\sigma)]\,d\sigma\,, \tag{73}$$

with

$$F^*(\eta^*)=\frac{1+\frac{1}{2}\eta^*-\frac{1}{10}\eta^{*2}+\frac{6}{105}\left(1-\frac{\delta}{S_0}\right)\left[(1+\eta^*)\frac{7}{S_0}m-3(1+\eta^*)^2 f^*-4f^*\right]}{\sqrt{1+\eta^*}}. \tag{74}$$

This function $F^*(\eta^*)$ generalizes the function $F(\eta)$ defined in (42) and is likewise close to unity for values of η^* likely to occur in the terrestrial planets. The boundary relationship generalizing eq. (45) is

$$f^*(1)=\frac{\frac{5}{2}m+\frac{10}{21}m^2+\frac{4}{7}f^*(1)\left[f^*(1)-\frac{3}{2}m\right]}{\eta^*(1)+2}. \tag{45$'$}$$

3.3 Third-order internal gravity field theory

The third-order theory (e.g. Denis 1985) is embodied in the system of figure equations (13) with $N=3$. All the necessary information is contained in the figure functions $s_2(s)$, $s_4(s)$ and $s_6(s)$, which are obtained by solving the following system of figure equations, shown here for reference purposes; details are given by Denis (1985):

$$F_2(\sigma)=:\left(-s_2+\frac{2}{7}s_2{}^2-\frac{29}{35}s_2{}^3+\frac{4}{7}s_2s_4\right)S_0+\left(1-\frac{6}{7}s_2+\frac{111}{35}s_2{}^2-\frac{6}{7}s_4\right)S_2+\left(1+\frac{4}{7}s_2+\frac{1}{35}s_2{}^2+\frac{4}{7}s_4\right)T_2$$
$$-\frac{10}{7}s_2S_4+\frac{8}{7}s_2T_4-\left(1-\frac{10}{7}s_2-\frac{9}{35}s_2{}^2+\frac{4}{7}s_4\right)\frac{m}{3}=0, \tag{75a}$$

$$F_4(\sigma)=:\left(-s_4+\frac{18}{35}s_2{}^2+\frac{40}{77}s_2s_4-\frac{108}{385}s_2{}^3\right)S_0+\left(-\frac{54}{35}s_2+\frac{648}{385}s_2{}^2-\frac{60}{77}s_4\right)S_2+\frac{4}{7}\left(\frac{9}{5}s_2+\frac{27}{55}s_2{}^2+\frac{10}{11}s_4\right)T_2$$
$$+\left(1-\frac{100}{77}s_2\right)S_4+\left(1+\frac{80}{77}s_2\right)T_4-\frac{12}{35}m\left(s_2-\frac{5}{22}s_2{}^2-\frac{96}{66}s_4\right)=0, \tag{75b}$$

$$F_6(\sigma)=:\left(-s_6+\frac{10}{11}s_2s_4-\frac{18}{77}s_2{}^3\right)S_0+\left(-\frac{15}{11}s_4+\frac{108}{77}s_2{}^2\right)S_2+\left(\frac{10}{11}s_4+\frac{18}{77}s_2{}^2\right)T_2-\frac{25}{11}s_2S_4$$
$$+\frac{20}{11}s_2T_4+S_6+T_6-\frac{10}{33}m\left(s_4+\frac{9}{35}s_2{}^2\right)=0. \tag{75c}$$

In the third-order approximation, the integral quantities S_2, T_2, S_4, T_4, S_6, T_6 are defined as follows:

$$S_2(\sigma)=\frac{3}{5\sigma^5}\int_0^\sigma \delta\frac{d}{d\sigma}\left[\sigma^5\left(s_2+\frac{4}{7}s_2{}^2+\frac{2}{35}s_2{}^3+\frac{8}{7}s_2s_4\right)\right]d\sigma, \tag{76}$$

$$T_2(\sigma)=\frac{3}{5}\int_\sigma^1 \delta\frac{d}{d\sigma}\left(s_2-\frac{1}{7}s_2{}^2+\frac{12}{35}s_2{}^3-\frac{2}{7}s_2s_4\right)d\sigma, \tag{77}$$

$$S_4(\sigma)=\frac{1}{3\sigma^7}\int_0^\sigma \delta\frac{d}{d\sigma}\left[\sigma^7\left(s_4+\frac{54}{35}s_2{}^2+\frac{120}{77}s_2s_4+\frac{108}{77}s_2{}^3\right)\right]d\sigma, \tag{78}$$

$$T_4(\sigma)=\frac{\sigma^2}{3}\int_\sigma^1 \delta\frac{d}{d\sigma}\left[\sigma^{-2}\left(s_4-\frac{27}{35}s_2{}^2-\frac{60}{77}s_2s_4+\frac{216}{385}s_2{}^3\right)\right]d\sigma, \tag{79}$$

$$S_6(\sigma)=\frac{3}{13\sigma^9}\int_0^\sigma \delta\frac{d}{d\sigma}\left[\sigma^9\left(s_6+\frac{40}{11}s_2s_4+\frac{24}{11}s_2{}^3\right)\right]d\sigma, \tag{80}$$

$$T_6(\sigma)=\frac{3\sigma^4}{13}\int_\sigma^1 \delta\frac{d}{d\sigma}\left[\sigma^{-4}\left(s_6-\frac{25}{11}s_2s_4+\frac{90}{77}s_2{}^3\right)\right]d\sigma. \tag{81}$$

and the coupling equation (7) becomes

$$s_0=-\frac{1}{5}s_2{}^2-\frac{2}{105}s_2{}^3. \tag{82}$$

The geometric oblateness f defined in (4) to $o(m^2)$ becomes to $o(m^3)$

$$f=-\frac{3}{2}s_2-\frac{3}{4}s_2{}^2-\frac{5}{8}s_4-\frac{21}{16}s_6+\frac{1}{4}s_2s_4-\frac{27}{40}s_2{}^3, \tag{83}$$

and the potential of any level surface is

$$U(\sigma)=-\frac{4\pi G\bar{\rho}R^2}{3}\,\sigma^2F_0(\sigma)\,, \tag{84a}$$

where

$$F_0(\sigma)=\left(1+\frac{2}{5}\,{s_2}^2-\frac{4}{105}\,{s_2}^3\right)S_0+T_0+\frac{3}{5}\left(-s_2+\frac{4}{7}\,{s_2}^2\right)S_2+\frac{2}{5}\left(s_2+\frac{1}{7}\,{s_2}^2\right)T_2+\frac{m}{3}\left(1-\frac{2}{5}\,s_2-\frac{9}{35}\,{s_2}^2\right) \tag{84b}$$

and

$$T_0(\sigma)=\frac{3}{2\sigma^2}\int_\sigma^1\delta\,\frac{d}{d\sigma}\left[\sigma^2\left(1-\frac{1}{5}\,{s_2}^2-\frac{4}{105}\,{s_2}^3\right)\right]d\sigma\,. \tag{84c}$$

3.3.1 *Third-order theory: integral approach*

The figure functions $s_2(\sigma)$, $s_4(\sigma)$ and $s_6(\sigma)$ are obtained from the following iteration scheme, by putting successively $k=0, 1, 2, \ldots$:

$$s_2^{(k+1)}=G_2(m,\,s_2^{(k)},\,s_4^{(k)},\,s_6^{(k)})\,, \tag{85a}$$

$$s_4^{(k+1)}=G_2(m,\,s_2^{(k+1)},\,s_4^{(k)},\,s_6^{(k)})\,, \tag{85b}$$

$$s_6^{(k+1)}=G_2,\,s_2^{(k+1)},\,s_4^{(k+1)},\,s_6^{(k)})\,. \tag{85c}$$

As before, the iteration may either be started from scratch, i.e. with

$$s_2^{(0)}=s_4^{(0)}=s_6^{(0)}=0\,,$$

or it may be started with $s_2^{(0)}$ and $s_4^{(0)}$ obtained using the second-order approximation. Again, we found that the overhead coding involved in the second alternative is too heavy to justify the slightly faster convergence of the iteration procedure. Iterations are stopped after the Kth integration such that

$$|s_{2n}^{(K)}(\sigma)-s_{2n}^{(K-1)}(\sigma)|\le\epsilon$$

for every value σ belonging to the interval [0, 1] and for $n=1$, 2, 3. An adequate tolerance for the Earth with its present-day spinning rate is obviously $\epsilon=(0.003)^3/10\approx3\times10^{-9}$.

Fig. 2 shows plots of the figure functions s_0, s_2, s_4 and s_6. These functions were computed by means of the iteration procedure (85) and formula (83) for the Earth models PREM (Dziewonski & Anderson 1981), PREMM and CGGM (Denis *et al.* 1997).

3.3.2 *Third-order theory: generalization of Darwin's approach*

The equation of an ellipsoid of revolution with equatorial radius a and flattening f expanded in f to $o(m^3)$ is

$$r(\theta)=a\left[1-f\cos^2\theta-\frac{3}{8}f^2\sin^2 2\theta+\frac{1}{8}f^3(1-5\sin^2\theta)\sin^2 2\theta\right]. \tag{86}$$

We seek, in accordance with Zharkov & Trubitsyn (1978, p. 244), the shape of the rotationally distorted body of the form

$$r(\theta)=a\left[1-f\cos^2\theta-\left(\frac{3}{8}f^2+\kappa\right)\sin^2 2\theta+\left(\frac{1}{8}f^3+\frac{\chi}{4}\right)(1-5\sin^2\theta)\sin^2 2\theta\right]. \tag{86$'$}$$

The correction functions $\kappa(\sigma)$ and $\chi(\sigma)$ express departures respectively of $o(m^2)$ and $o(m^3)$ of the level surfaces from exact ellipsoids. After lengthy but straightforward calculations we find

$$\kappa=\frac{35}{32}\,s_4-\frac{27}{32}\,{s_2}^2-\frac{27}{32}\,{s_2}^3-\frac{5}{32}\,s_2s_4+\frac{63}{40}\,s_6\,, \tag{87a}$$

$$\chi=\frac{27}{16}\,{s_2}^3-\frac{231}{80}\,s_6\,, \tag{87b}$$

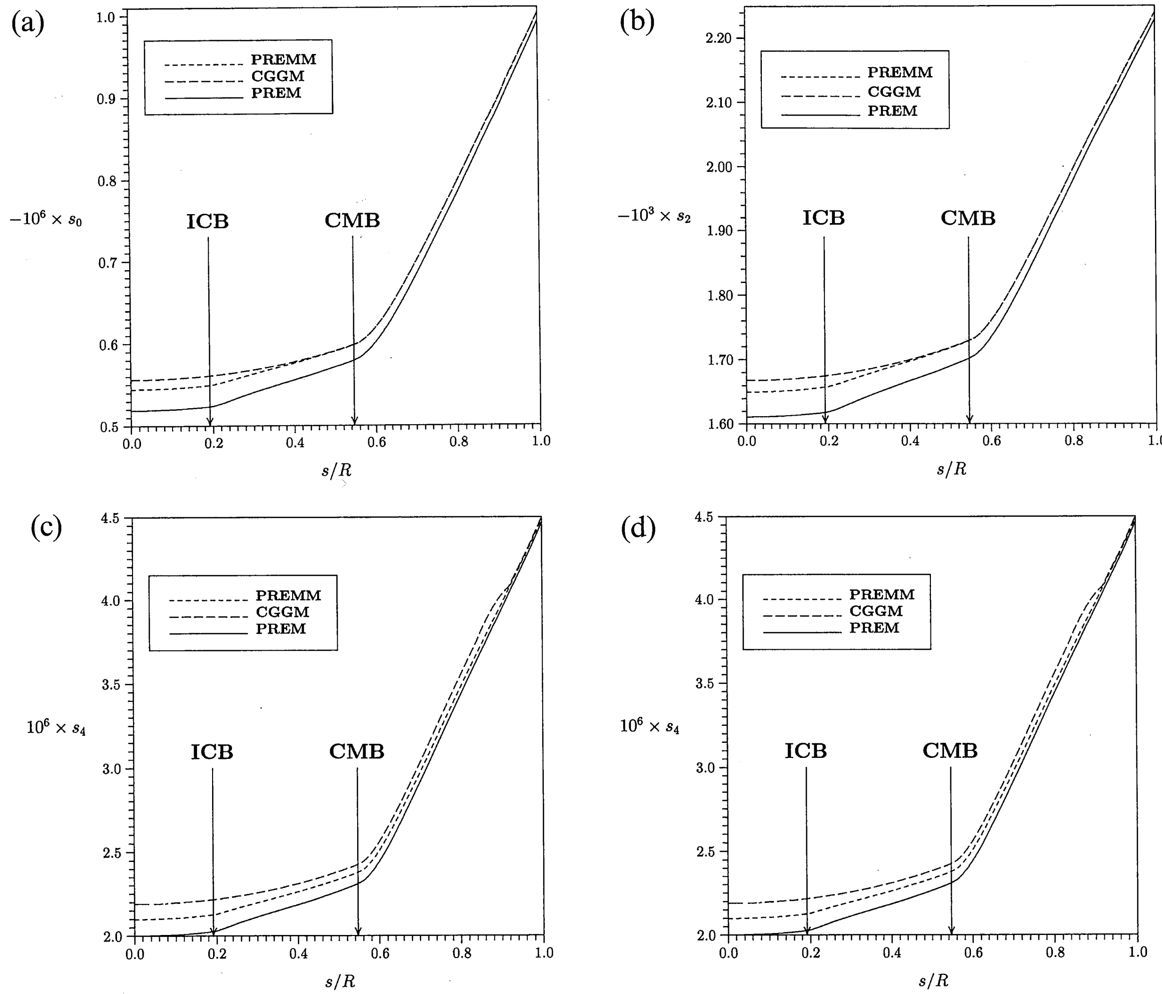

Figure 2. Plots of the figure functions s_0, s_2, s_4 and s_6 computed by means of the third-order integral approach, for the Earth models PREM (Dziewonski & Anderson 1981), CGGM and PREMM (Denis *et al.* 1997). The figure function s_0 can be obtained from s_2 by means of the coupling equation (7). The figure functions s_2, s_4, s_6 represent the only information necessary to deduce any parameter characterizing the hydrostatic shape of the Earth in a third-order approximation.

or, reversely,

$$s_2=-\frac{2}{3}f-\frac{23}{63}f^2-\frac{8}{21}\,\kappa-\frac{4}{27}f^3+\frac{2}{21}\,\chi-\frac{152}{315}f\kappa\,, \tag{88a}$$

$$s_4=\frac{12}{35}f^2+\frac{32}{35}\,\kappa+\frac{4}{11}f^3+\frac{192}{385}\,\chi+\frac{32}{105}f\kappa\,, \tag{88b}$$

$$s_6=-\frac{40}{231}f^3-\frac{80}{231}\,\chi\,. \tag{88c}$$

In terms of the variables f, κ and χ, the figure equation $F_6(\sigma)=0$ becomes

$$\left(f^3+\frac{32}{7}f\kappa-\frac{10}{7}\,\chi\right)S_0+\frac{15}{14}(f^2+8\kappa)S_2-\frac{25}{4}fS_4+5fT_4=\frac{33}{8}\,(S_6+T_6)\,. \tag{89}$$

In the latter integro-differential equation, we notice that f and S_2 need to be evaluated to o(m), which can be done using Clairaut's first-order equation. The quantities κ, S_4 and T_4 need to be evaluated to o(m^2), which can be done by solving Darwin's second-order equation. Finally, χ, S_6 and T_6 must be evaluated to o(m^3). The expressions of S_6 and T_6 to o(m^3) in the new variables are

$$S_6(\sigma)=-\frac{8}{21\sigma^9}\int_0^\sigma \delta\,\frac{d}{d\sigma}\left[\sigma^9\left(f^3+\frac{192}{143}f\kappa+\frac{30}{143}\,\chi\right)\right]d\sigma\,, \tag{90}$$

$$T_6(\sigma)=-\frac{80\sigma^4}{1001}\int_\sigma^1 \delta\,\frac{d}{d\sigma}[\sigma^{-4}(\chi-4f\kappa)]\,d\sigma\,. \tag{91}$$

Differentiating twice eq. (89) with respect to σ leads to a second-order ODE for χ, which we will not elaborate on here. The solution of the latter provides $\chi(\sigma)$. Thus, the differential procedure in the third-order approximation is quite similar to that explained for the second-order approximation, except that it involves one more step and some more coding.

4 SHAPE OF THE EARTH DERIVED FROM THE DEFORMATION EQUATIONS

Two alternative ways of computing the geometric oblateness of the Earth have recently been investigated by Denis & Rogister (1995) and Rogister & Denis (1995). They consist of solving the small deformation equations of a spherical body subjected to a centrifugal force. Following Denis (1993), let us sketch the derivation of these equations, and put them in a form similar to that originally given by Pekeris & Jarosch (1958) and Alterman, Jarosch & Pekeris (1959). Closely related forms have been considered by Takeuchi (1950), Molodensky (1961) and others (e.g. Longman 1962; Jeffreys & Vicente 1966; Wiggins 1968; Takeuchi & Saito 1972; Denis 1974; Chapman & Woodhouse 1981). These equations express explicitly the conservation of mass and linear momentum.

4.1 Equations governing small elastic deformations of a spherical Earth model

Consider an undisturbed spherically symmetric, non-rotating, linearly elastic and isotropic body. Such a SNREI mechanical Earth model (Dahlen 1968) may be represented in terms of three functions of radius r, which we choose here to be the mass density ρ_o, and the two Lamé coefficients λ_o and μ_o. For this spherical reference model, we assume that the equation of hydrostatic equilibrium holds, namely

$$\frac{dp_o}{dr} = -\rho_o \frac{d\Phi_o}{dr} . \tag{92a}$$

$p_o(r)$ is the undisturbed hydrostatic (or lithostatic) pressure and Φ_o is the undisturbed gravity potential, which may be computed from $\rho_o(r)$ by means of the undisturbed Poisson equation

$$\frac{d^2\Phi_o}{dr^2} + \frac{2}{r}\frac{d\Phi_o}{dr} = 4\pi G \rho_o . \tag{92b}$$

In the disturbed state, mass density ρ, stress tensor $\boldsymbol{T}$, and gravity potential Φ become in a local (Eulerian) description

$$\rho(r, \theta, \varphi) = \rho_o(r) + \rho_1(r, \theta, \varphi) , \tag{93a}$$

$$\boldsymbol{T}(r, \theta, \varphi) = -p_o(r)\boldsymbol{E} + \boldsymbol{T}_1(r, \theta, \varphi) , \tag{93b}$$

$$\Phi(r, \theta, \varphi) = \Phi_o(r) + \Phi_1(r, \theta, \varphi) . \tag{93c}$$

An arbitrary space point P associated with the position vector $\mathbf{r}$ is expressed here in spherical coordinates (r, θ, φ), where r is the central distance, θ is the colatitude and φ is the longitude. $\boldsymbol{E}$ is the undisturbed metric tensor, that is the unit matrix in our context. The physical meaning of the other quantities should be clear. Indeed, in the undisturbed state, density, stress and gravity potential are constant on the spherical surface $r(\theta, \varphi) = s$, where $s = |\mathbf{r}|$. Thus, ρ_1, $\boldsymbol{T}_1$, Φ_1 in fact represent the Eulerian perturbations $\partial\rho$, $\partial\boldsymbol{T}$, $\partial\Phi$ of density, stress and gravity potential, respectively, considered at the space point $\mathbf{r}$ on the spherical surface $r(\theta, \varphi) = s$. The variations in the physical attributes at point $\mathbf{r}$ at a given instant t are brought about by the particular mass point $\mathbf{X}$ situated in the disturbed state at point $\mathbf{r} = \mathbf{X} + \mathbf{u}$, where $\mathbf{u}(\mathbf{X}, t)$ denotes the displacement vector of mass point $\mathbf{X}$ from its original position.

We deal here with small deviations from the equilibrium values, in order to neglect squares, cross-products and *a fortiori* higher powers of the particle displacements and the ensuing perturbations in physical properties. This makes the system of equations that describe the deforming motion linear. In the particular application we shall use these equations for in the present paper, the displacements considered are proportional to the geodynamic constant m, and thus the algorithms described in this section correspond essentially to first-order figure theory. The linearized momentum and Poisson equations are respectively

$$\frac{\partial^2\mathbf{u}}{\partial t^2} = -\nabla(\Phi_1 + \mathbf{u}\cdot\nabla\Phi_o) + (\nabla\Phi_o)\nabla\cdot\mathbf{u} + \frac{\nabla\cdot\boldsymbol{T}_1^{\mathrm{L}}(\mathbf{X}, t) + \mathbf{f}}{\rho_o} \tag{94}$$

and

$$\nabla^2\Phi_1 = -4\pi G \nabla\cdot(\rho_o\mathbf{u}) . \tag{95}$$

Note that in the momentum equation we have explicitly introduced the Lagrangian incremental stress tensor $\delta\boldsymbol{T} = \boldsymbol{T}_1^{\mathrm{L}}(\mathbf{X}, t)$, instead of the Eulerian stress perturbation $\partial\boldsymbol{T} = \boldsymbol{T}_1(\mathbf{r}, t)$. These quantities are related to each other by $\delta\boldsymbol{T} = \partial\boldsymbol{T} + \mathbf{u}\cdot\nabla\boldsymbol{T}$, i.e.

$$\boldsymbol{T}_1^{\mathrm{L}}(\mathbf{X}, t) \approx \boldsymbol{T}_1(\mathbf{r}, t) - \mathbf{u}\cdot\nabla p_o\boldsymbol{E} . \tag{96}$$

Because we consider here a linear isotropic elastic deformation, Hooke's law applies for the incremental stress carried by any particle $\mathbf{X}$ under the form

$$\boldsymbol{T}_1^{\mathrm{L}}(\mathbf{X}, t) = \lambda_o \nabla\cdot\mathbf{u}\,\boldsymbol{E} + \mu_o[\nabla\mathbf{u} + \widetilde{\nabla\mathbf{u}}], \tag{97}$$

where a tilde denotes transposition. Using spherical coordinates, and writing

$$\mathbf{u} = \frac{\mathbf{r}}{r}\,U + \nabla\times(W\mathbf{r}) + \frac{\mathbf{r}}{r}\times\nabla\times(V\mathbf{r}), \tag{98}$$

$$\boldsymbol{T}_1^{\mathrm{L}}(\mathbf{X}, t)\cdot\frac{\mathbf{r}}{r} = \frac{\mathbf{r}}{r}\,R + \nabla\times(T\mathbf{r}) + \frac{\mathbf{r}}{r}\times\nabla\times(S\mathbf{r}), \tag{99}$$

we expand the scalar fields U, V, W, R, S, T, as well as the gravity potential perturbation $\Phi_1 = -P$, the dilatation $X = \nabla\cdot\mathbf{u}$, and a disturbing scalar load potential Z into spherical harmonics. Thus, for any of these fields denoted by F, we write

$$F(r, \theta, \varphi, t) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} F_\ell^m(r)\,Y_\ell^m(\theta, \varphi)\exp(-i\omega t), \tag{100}$$

where $Y_\ell^m(\theta, \varphi)$ is a spherical surface harmonic of degree ℓ and order m.

Making the notations less cumbersome by dropping the indices ℓ and m from the generic quantities $F_\ell^m(r)$ and omitting the common factor $Y_\ell^m(\theta, \varphi)\exp(-i\omega t)$, we obtain the sixth-order spheroidal system

$$\omega^2 U + P' + g_o X - (g_o U)' + (\lambda_o X + 2\mu_o U')'\rho_o^{-1} + [4rU' - 4U + \ell(\ell+1)(3V - U - rV')]\mu_o\rho_o^{-1}r^{-2} = Z', \tag{101a}$$

$$\omega^2 V + Pr^{-1} - g_o U r^{-1} + \{\lambda_o X r^{-1} + [(V' - Vr^{-1} + Ur^{-1})\mu_o]'\}\rho_o^{-1} + [5U + 3rV' - V - 2\ell(\ell+1)V]\mu_o\rho_o^{-1}r^{-2} = Zr^{-1}, \tag{101b}$$

$$P'' + 2r^{-1}P' - \ell(\ell+1)r^{-2}P = 4\pi G(\rho_o' U + \rho_o X), \tag{101c}$$

$$X = U' + 2r^{-1}U - \ell(\ell+1)r^{-1}V, \tag{101d}$$

as well as the second-order toroidal system

$$\omega^2 W + [W'' + 2r^{-1}W' - \ell(\ell+1)r^{-2}W]\mu_o\rho_o^{-1} + (W' - Wr^{-1})\mu_o'\rho_o^{-1} = 0. \tag{102}$$

A prime denotes derivation with respect to r, and the undisturbed gravity is $g_o = \Phi_o'$.

4.2 Physical components of displacement and stress

According to eqs (98) and (97), the physical components of the spheroidal (σ) and toroidal (τ) displacement and stress fields, for a given surface harmonic of degree ℓ and order m, are respectively

$$u_r^{(\sigma)} = U(r)Y_\ell^m(\theta, \varphi), \qquad u_\theta^{(\sigma)} = V(r)\,\frac{\partial Y_\ell^m(\theta, \varphi)}{\partial\theta}, \qquad u_\varphi^{(\sigma)} = V(r)\,\frac{\partial Y_\ell^m(\theta, \varphi)}{\sin\theta\partial\varphi}, \tag{103}$$

$$u_r^{(\tau)} = 0, \qquad u_\theta^{(\tau)} = W(r)\,\frac{\partial Y_\ell^m(\theta, \varphi)}{\sin\theta\partial\varphi}, \qquad u_\varphi^{(\tau)} = -W(r)\,\frac{\partial Y_\ell^m(\theta, \varphi)}{\partial\theta}, \tag{104}$$

$$T_{rr}^{(\sigma)} = R_\ell^m(r)Y_\ell^m, \qquad T_{r\theta}^{(\sigma)} = S_\ell^m(r)\,\frac{\partial Y_\ell^m}{\partial\theta}, \qquad T_{r\varphi}^{(\sigma)} = S_\ell^m(r)\,\frac{\partial Y_\ell^m}{\sin\theta\partial\varphi}, \tag{105}$$

$$T_{\theta\theta}^{(\sigma)} = \left\{\lambda_o U' + \frac{\lambda_o + 2\mu_o}{r}[2U - \ell(\ell+1)V] - \frac{2\mu_o U}{r}\right\}Y_\ell^m - \frac{2\mu_o V}{r}\,\frac{\partial^2 Y_\ell^m}{\partial\theta^2}, \tag{106}$$

$$T_{\varphi\varphi}^{(\sigma)} = \left\{\lambda_o U' + \frac{\lambda_o + 2\mu_o}{r}[2U - \ell(\ell+1)V] - \frac{2\mu_o U}{r}\right\}Y_\ell^m - \frac{2\mu_o V}{r}\left(\cot\theta\,\frac{\partial Y_\ell^m}{\partial\theta} + \frac{\partial^2 Y_\ell^m}{\sin^2\theta\partial\varphi^2}\right), \tag{107}$$

$$T_{\theta\varphi}^{(\sigma)} = \frac{2\mu_o V}{r}\left(\frac{\partial^2 Y_\ell^m}{\sin\theta\partial\theta\partial\varphi} - \frac{\cot\theta\partial Y_\ell^m}{\sin\theta\partial\varphi}\right), \tag{108}$$

$$T_{rr}^{(\tau)} = 0, \qquad T_{r\theta}^{(\tau)} = T_\ell^m(r)\,\frac{\partial Y_\ell^m}{\sin\theta\partial\varphi}, \qquad T_{r\varphi}^{(\tau)} = -T_\ell^m(r)\,\frac{\partial Y_\ell^m}{\partial\theta}, \tag{109}$$

$$T_{\theta\theta}^{(\tau)} = -T_{\varphi\varphi}^{(\tau)} = \frac{2\mu_o W}{r}\left(\frac{\partial^2 Y_\ell^m}{\sin\theta\partial\theta\partial\varphi} - \frac{\cot\theta\partial Y_\ell^m}{\sin\theta\partial\varphi}\right), \tag{110}$$

$$T_{\theta\varphi}^{(\tau)} = \frac{\mu_o W}{r}\left(-\frac{\partial^2 Y_\ell^m}{\partial\theta^2} + \frac{\cot\theta\partial Y_\ell^m}{\partial\theta} + \frac{\partial^2 Y_\ell^m}{\sin^2\theta\partial\varphi^2}\right). \tag{111}$$

The radial factors $R_\ell^m(r)$, $S_\ell^m(r)$, $T_\ell^m(r)$ of the scalars associated with the normal stress components, which we abbreviate to $R(r)$, $S(r)$, $T(r)$ respectively, are

$$R=(\lambda_o+2\mu_o)U'+\frac{\lambda_o}{r}[2U-\ell(\ell+1)V]=\lambda_o X+2\mu_o U', \tag{112}$$

$$S=\mu_o\left(V'-\frac{V}{r}+\frac{U}{r}\right), \tag{113}$$

$$T=\mu_o\left(W'-\frac{W}{r}\right). \tag{114}$$

4.3 Boundary conditions

Any physically acceptable solution of the system of ordinary differential equations (101–102) must satisfy specific boundary conditions at borders between two different continua. Both kinematic and dynamic boundary conditions ought to be fulfilled.

Let P be an arbitrary material point on a border surface, with position vector $\mathbf{X}$ in the unstrained body, with position vector $\mathbf{r}$ in the strained body, and let its distance to the Earth's centre be r. Moreover, define for any quantity $F(r)$

$$[F(r)]_-^+=F(r+\varepsilon)-F(r-\varepsilon) \qquad \text{for } \varepsilon\to 0. \tag{115}$$

Solid–solid borders such as the crust–mantle boundary are generally modelled as slip-free interfaces, namely $[U(r)]_-^+=[V(r)]_-^+=[W(r)]_-^+=0$. Fluid–fluid borders, if of any concern, should be modelled as slip-free interfaces too. Solid–fluid or fluid–solid borders such as the ICB, the CMB and the crust–ocean boundary (COB) are usually modelled as slip interfaces: $[U(r)]_-^+=0$, $[V(r)]_-^+\neq 0$, $[W(r)]_-^+\neq 0$. Those are the kinematic boundary conditions. As far as the dynamic boundary conditions are concerned, it is common practice to assume that the normal components of the *elastic* stress tensor on a border surface are continuous, i.e. $[R(r)]_-^+=[S(r)]_-^+=[T(r)]_-^+=0$.

We must also stipulate the continuity across any boundary of the perturbations of the gravity work function P, and of the gravity function Ψ itself, i.e. $[P(r)]_-^+=[\Psi(r)]_-^+=0$. We obtain the expression of Ψ by noting that the first-order perturbation of gravity is the result of two effects: an integrated effect stemming from the perturbation of the gravity potential ($\to\Phi_1'=-P'$), and a local effect due to the displacement of the mass point with respect to the equilibrium configuration ($\to +4\pi G\rho_o U$). The latter may be envisaged as the jump occurring in the normal derivative of the potential of a surface layer of surface mass density $\rho_o U$. Thus,

$$\Psi=-P'+4\pi G\rho_o U. \tag{116}$$

All these boundary conditions hold of course on the border surfaces in the strained body, *not* in the unstrained body. However, because we assume the deformed borders to remain all the time very close to the initial borders, we may actually consider the boundary conditions in the unstrained body, because differences such as $U(\mathbf{r})-U(\mathbf{X})$, $V(\mathbf{r})-V(\mathbf{X})$, etc., are quantities of the second order of smallness and may be neglected in this linear theory.

Let us disregard for the moment the presence of matter outside the Earth's outer surface. Under these circumstances, if we match at $r=a$ the internal potential field, determined by Poisson's equation, with the external potential field, determined by Laplace's equation, and the internal gravity field with the external gravity field, we arrive at the following boundary condition:

$$\Psi(a)=\frac{\ell+1}{a}P(a). \tag{117}$$

For convenience we define the auxiliary function

$$Q(r)=\Psi(r)-\frac{\ell+1}{r}P(r), \tag{118}$$

which is continuous everywhere. Then the boundary condition (117) takes the shorter form

$$Q(a)=0. \tag{117'}$$

If, moreover, we assume the outer surface to be free from normal stress, we also have

$$R(a)=S(a)=T(a)=0. \tag{119}$$

We notice that the spheroidal differential system (101) is uncoupled from the toroidal differential system (102). The boundary conditions considered here do not give rise to any coupling between spheroidal and toroidal variables either. Thus, the spheroidal and toroidal problems may be considered in this case independently. The uncoupling of spheroidal and toroidal deformation is due to spherical symmetry and, in physics, is called *Coulomb degeneracy*. The rotational load problem we are dealing with does not involve the toroidal system. Therefore, we shall forget about eq. (102) and the associated boundary conditions from now on. We wish to mention only that the toroidal loading has been considered in the context of geodynamics, for example by Varga (1992).

4.4 Some cases of spheroidal deformation

As far as spheroidal deformation is concerned, we mention the following cases, where for reasons of clarity we reintroduce the indices ℓ and m and put $m=0$.

(1) *Free oscillations* are computed after putting $Z_\ell^0(r)=0$ and then integrating the resulting homogeneous system (101) with the homogeneous boundary conditions $R_\ell^0(a)=S_\ell^0(a)=Q_\ell^0(a)=0$.

(2) *Tidal deformation* is dealt with by noting that the tidal potential, described by $Z_\ell^0(r)$ in eqs (101a,b), is harmonic everywhere outside the attracting cosmic body. Thus, we may replace in eqs (101a,b,c) the variable P_ℓ^0 with the new variable $P_\ell^0-Z_\ell^0$, renamed P_ℓ^0, and replace the right-hand sides of eqs (101a,b) with zeros. Thus, the system (101) is actually transformed into the same homogeneous system as that providing the free oscillations. The non-homogeneous character of the tidal loading problem occurs in the boundary conditions to be applied at the outer boundary, which now become

$$R_\ell^0(a)=0\,,\qquad S_\ell^0(a)=0\,,\qquad Q_\ell^0(a)=\frac{2\ell+1}{a}\,Z_\ell^0(a)\,. \tag{120}$$

Integration provides in particular the values $U_\ell^0(a)$, $V_\ell^0(a)$ and $\Phi_\ell^0(a)=-P_\ell^0(a)$, in terms of which we obtain the tidal Shida–Love numbers (defined for $\ell\geq 2$)

$$h_\ell=-g_o(a)\,\frac{U_\ell^0(a)}{Z_\ell^0(a)}\,,\qquad k_\ell=\frac{\Phi_\ell^0(a)}{Z_\ell^0(a)}-1\,,\qquad l_\ell=-g_o(a)\,\frac{V_\ell^0(a)}{Z_\ell^0(a)}\,. \tag{121}$$

(3) *Deformation under surface mass loads* has been thoroughly investigated by Longman (1962, 1963) and Farrell (1972). In this case, the disturbing potential Z_ℓ^0 in (101a,b) is harmonic everywhere except on the outer boundary $r=a$. Thus, load deformation can be computed using exactly the same algorithm as tidal deformation, provided we now apply the following boundary conditions:

$$R_\ell^0(a)=\frac{2\ell+1}{3}\,\bar{\rho}Z_\ell^0(a)\,,\qquad S_\ell^0(a)=0\,,\qquad Q_\ell^0(a)=\frac{2\ell+1}{a}\,Z_\ell^0(a)\,. \tag{122}$$

The surface mass load Longman–Love numbers are defined in the same way as the tidal Shida–Love numbers, namely (for $\ell\geq 0$)

$$h'_\ell=-g_o(a)\,\frac{U_\ell^0(a)}{Z_\ell^0(a)}\,,\qquad k'_\ell=\frac{\Phi_\ell^0(a)}{Z_\ell^0(a)}-1\,,\qquad l'_\ell=-g_o(a)\,\frac{V_\ell^0(a)}{Z_\ell^0(a)}\,. \tag{123}$$

(4) *Rotational deformation* can be computed in the same way as tidal deformation, because the centrifugal potential $-\frac{1}{3}\Omega^2r^2[1-P_2(\cos\theta)]$ is again harmonic everywhere. It gives rise to rotational Love numbers of harmonic degrees 0 and 2, defined again in a way similar to (121) or (123):

$$h''_0=-g_o(a)\,\frac{U_0^0(a)}{Z_0^0(a)}\,,\qquad k''_0=\frac{\Phi_0^0(a)}{Z_0^0(a)}-1\,,\qquad l''_0=-g_o(a)\,\frac{V_0^0(a)}{Z_0^0(a)}\,, \tag{124}$$

$$h''_2=-g_o(a)\,\frac{U_2^0(a)}{Z_2^0(a)}\,,\qquad k''_2=\frac{\Phi_2^0(a)}{Z_2^0(a)}-1\,,\qquad l''_2=-g_o(a)\,\frac{V_2^0(a)}{Z_2^0(a)}\,. \tag{125}$$

Numerically, h''_2 is of course equal to the tidal Love number h_2.

4.5 Rotational distortion of incompressible spherical models

Let us assume that an incompressible ($\lambda_o=\infty$), solid ($0<\mu_o<\infty$), uniform ($\rho'_o=\mu'_o=0$) spherical layer undergoes spheroidal deformation. In this idealized case, the deformation of the layer must be incompressive, i.e. $X(r)=0$, for the quantity $\Pi(r)=\lambda_oX$ occurring in eqs (101a) and (101b) to remain finite. Love (1911) calls this quantity $\Pi(r)$ an *additional pressure*. Thus, eqs (101a), (101b), (101c) and (101d) become respectively in the given layer

$$\omega^2U+P'-(g_oU)'+\rho_o^{-1}\Pi'+2\mu_o\rho_o^{-1}U''+\mu_o\rho_or^{-1}[4(U'-Ur^{-1})+\ell(\ell+1)(3Vr^{-1}-Ur^{-1}-V')]=Z'\,, \tag{126a}$$

$$\omega^2V+Pr^{-1}-g_oUr^{-1}+\rho_o^{-1}\{\Pi r^{-1}+[\mu_o(V'-Vr^{-1}+Ur^{-1})]'\}+\mu_o\rho_o^{-1}r^{-1}[5Ur^{-1}+3V'-Vr^{-1}-2\ell(\ell+1)Vr^{-1}]=Zr^{-1}\,, \tag{126b}$$

$$P''+2r^{-1}P'-\ell(\ell+1)r^{-2}P=0\,, \tag{126c}$$

$$U'+2r^{-1}U-\ell(\ell+1)r^{-1}V=0\,. \tag{126d}$$

If we multiply eq. (126b) by r, then differentiate the result of this multiplication with respect to r, subtract the new result from eq. (126a), and take into account eq. (126d), we obtain finally a fourth-order ODE for the variable U, viz.

$$\omega^2\left[U-\frac{r^2U''+4rU'+2U}{\ell(\ell+1)}\right]-\frac{\mu_o}{\rho_o}\left[\frac{r^2U''''+8rU'''+12U''}{\ell(\ell+1)}-2U''-\frac{4U'}{r}-\frac{2U}{r^2}+\frac{\ell(\ell+1)U}{r^2}\right]=0\,. \tag{127}$$

For $l \geq 1$, the general solution of eq. (127) is

$$U(r)=A_1 r^{\ell-1}+A_2\frac{j_\ell(kr)}{r}+A_3 r^{-\ell-2}+A_4\frac{j_{-\ell-1}(kr)}{r}\,, \tag{128a}$$

where $j_\ell(kr)$ and $j_{-\ell-1}(kr)$ are respectively the spherical Bessel function and the spherical Neumann function of order ℓ and argument kr, with $k^2=\rho_o\omega^2/\mu_o$. The general solution of eq. (126c) is the solid harmonic

$$P(r)=A_5 r^\ell+A_6 r^{-\ell-1}\,. \tag{128b}$$

The six constants $A_1, A_2, A_3, A_4, A_5, A_6$ must be determined in each spherical shell by means of adequate boundary conditions. In the central region, we have of course $A_3=A_4=A_6=0$.

In the static case, obtained by setting $\omega^2=0$ in the equations above, the general solution degenerates into

$$U(r)=A_1 r^{\ell-1}+A_2 r^{\ell+1}+A_3 r^{-\ell-2}+A_4 r^{-\ell}\,. \tag{129}$$

On the other hand, if we have to deal with fluid incompressible homogeneous spherical layers, we put $\mu_o=0$ in eqs (126a,b) to obtain

$$\omega^2 U+P'-(g_o U)'+\frac{\Pi'}{\rho_o}=Z'\,, \tag{130a}$$

$$\omega^2 rV+P-g_o U+\frac{\Pi}{\rho_o}=Z\,, \tag{130b}$$

where $\Pi=\lambda_o X$. Now differentiate eq. (130b) with respect to r, subtract the relation thus obtained from eq. (130a), assume that ω is not zero, and use the relation $X=0$ to obtain finally

$$U(r)=B_1 r^{\ell-1}+B_2 r^{-\ell-2}\,, \tag{131}$$

B_1 and B_2 being integration constants. Eqs (126c,d) remain of course valid. Therefore, the solution (128b) for P holds in fluid layers as well.

To obtain the solution (131), we had to exclude the static case, namely $\omega \neq 0$. Indeed, in that case, eq. (130a) is merely the derivative with respect to r of eq. (130b), and U cannot be computed unambiguously in a fluid layer (Rogister 1995). However, unlike the general case of a realistic compressible Earth model in which the outer core does not satisfy perfectly the Adams–Williamson condition (see below), here it is possible to obtain a static solution for the equipotential surfaces which fulfils all the boundary conditions. Moreover, it seems natural enough, for reasons of continuity of solutions in the limiting process $\omega^2 \to 0$, to choose the form (131) for $U(r)$ in the static case as well. $V(r)$ is then obtained by means of the relation $X(r)=0$, yielding

$$V(r)=B_1\frac{r^{\ell-1}}{\ell}-B_2\frac{r^{-\ell-2}}{\ell+1}\,. \tag{132}$$

More specifically, let us now consider rotational distortion produced by a change of the centrifugal potential $-(1/3)\Omega^2 r^2[1-P_2(\cos\theta)]$, possibly modulated in time with an angular frequency ω. In the situation of incompressive deformation that we are dealing with in this paper, we are not concerned with the monopole term, and shall consider only the quadrupole source term

$$Z_2(r,\theta)=\frac{v\Omega^2}{3}\,r^2 P_2(\cos\theta)\,. \tag{133}$$

We are able to introduce the relative change v in the centrifugal potential instead of the potential itself because the equations are linear. Thus, if we set $v=1$ we obtain the shape of the Earth caused by its spinning rate Ω. On the other hand, if we wish to compute the *change of shape* caused by a change $\Delta\Omega$ of the rotation rate, we simply put $v=2\Delta\Omega/\Omega$. For a fluctuation ΔLOD of the length of day expressed in ms, we set $v \approx -2.3\times10^{-8}\,\Delta LOD$.

The flattening f (respectively the incremental flattening Δf if $v<1$) is computed by means of the formula

$$f=-\frac{3}{2}\,\frac{\Phi_2^0(r)+Z_2(r)}{rg_o(r)}\,. \tag{134}$$

It is important to notice that we are using here the flattening of the equipotential surfaces, *not* the flattening of the material surfaces given to first order by

$$f_{\rm mat}=-\frac{3}{2}\,\frac{U_2^0(r)}{r}\,. \tag{135}$$

The latter is quite distinct from the equipotential flattening in the solid bulk. Some numerical results will be given below.

In Fig. 3 plots are shown of the results for the reciprocal flattening f^{-1} of level surfaces obtained with our numeric code (named IncompDef), which computes deformations of incompressible spherical models. The latter follows essentially the mathematical formulation given above. We subdivided the piecewise continuous density distribution of PREM into a selectable number N of uniform layers, the thickness of each layer being loosely adapted to the local density gradient in PREM. In the runs illustrated in Fig. 3 we used $N=50$. The curve labelled 'hydrostatic' serves as a reference. It was obtained by means of another code (named Clairaut3) based on the Clairaut–Laplace–Lyapunov integro-differential approach sketched above (Section 3.3) and described in full detail by Denis (1985, 1989). The values of f^{-1} were obtained by means of (4) expanded to o(m^3) from the figure functions plotted in Fig. 2. Superimposed on this reference curve lies a curve obtained by means of the code IncompDef, where we put zero for the shear modulus in each of the 50 layers, and $\nu=1$. At the scale used for drawing Fig. 3, the slight differences of the two curves are barely noticeable.

This result shows that direct integration of the deformation equations leads to the same result as the first-order Clairaut theory, which uses the concept of level surfaces. However, the former method is more powerful if we are interested only in a first-order approximation, because it allows us to compute the flattening of rotationally distorted level and material surfaces in non-hydrostatic situations. Thus, the uppermost curve labelled 'elastic' shows the reciprocal flattening of the level surfaces in the rotationally distorted incompressible *elastic* PREM model, that is we kept the original shear modulus in the mantle and inner core of PREM, but took volume averages in each of the 50 layers. Obviously, in the elastic case the distorted material surfaces do not coincide with the distorted level surfaces. Moreover, f^{-1} takes on much larger values in this case than in the hydrostatic case. The fact that distortion is much smaller is due to the shear resistance in redistributing mass, giving rise to less pronounced changes of the gravity potential, and therefore to lesser flattening of the equipotentials.

If we decrease the shear modulus by multiplying it in each layer by some constant scale factor $\Sigma_\mu < 1$, the values of f^{-1} decrease and nearly coincide with the hydrostatic values of f^{-1} for $\Sigma_\mu \leq 0.001$. In Fig. 3 the curve corresponding to $\Sigma_\mu = 0.01$, shown as a long-dashed line, is still distinct from the hydrostatic reference curve, but we already notice a convergence towards the latter.

The two curves in Fig. 3 drawn as short-dashed lines represent interesting situations. They were both obtained for the same intermediate value of the scale factor, namely $\Sigma_\mu = 0.1$, with $\nu = 1$ as for all the curves in Fig. 3. The upper such curve, labelled

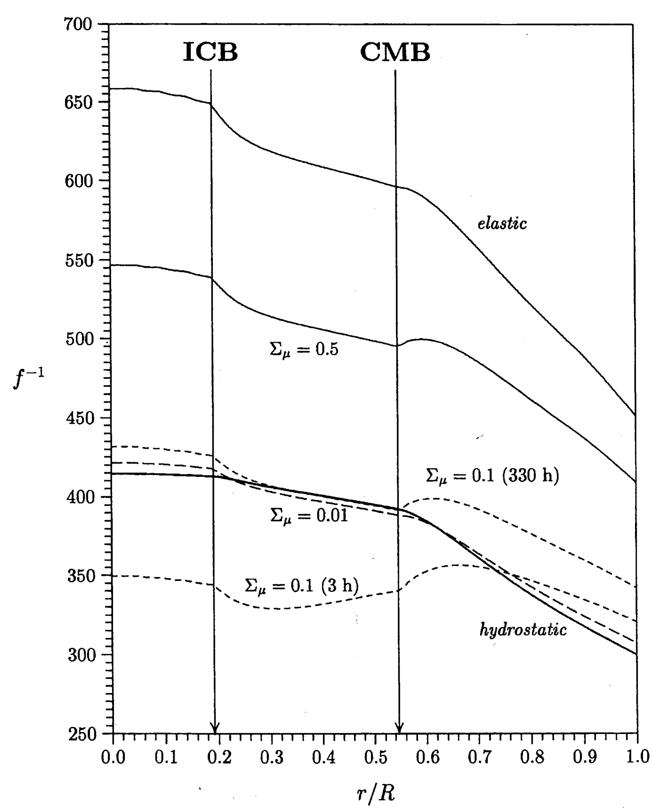

Figure 3. Inverse flattenings f^{-1} of equipotential surfaces as a function of the shear modulus. The basic density model is PREM, which has been subdivided into 50 uniform layers of equal thickness along the radius. In all cases the layers are considered incompressible by assuming the compressional-wave velocity infinite. In the case labelled 'elastic' (upper curve), essentially the original shear-wave velocity was retained (as far as 50 uniform layers can simulate the original distribution). In the case labelled 'hydrostatic', the shear velocity is zero everywhere. The other cases correspond to intermediate situations, where the original PREM shear modulus has everywhere been multiplied by the indicated value of Σ_μ.

'$\Sigma_\mu=0.1$ (330 h)', corresponds to the static case. It has actually been obtained for a forcing period of 330 hr (that is, close to the period of the fortnightly tide), but at the level of accuracy set by a first-order approximation it is numerically identical to the curve obtained in the truly static case. We checked this fact with our code IncompDef, which was devised to work both in strictly static (infinite forcing period) and dynamic (finite forcing period) circumstances. We notice that the curve has a minimum in the outer core, near the CMB, with values of f^{-1} which fall *below* the corresponding hydrostatic reference values. This behaviour can easily be explained if we realize that the shear modulus is small enough to allow relatively large motions to occur in the mantle, with large associated mass redistribution, the originally coinciding material and equipotential surfaces being quite distinct when rotation occurs. The lower dotted curve, labelled '$\Sigma_\mu=0.1$ (3 h)', corresponds to a dynamic case in which we considered a forcing period of 3 hr. The difference from the corresponding static curve is quite obvious. We should like to mention here that the forcing period of 330 hr was not chosen in a completely arbitrary way to represent pseudo-static deformation. In the next section we shall see that static deformation *cannot* be computed for realistic Earth models made up of compressible layers. In the case of PREM, a period of 330 hr is about the longest period our code can handle without running into numerical instability problems.

On the other hand, if we increase rigidity by multiplying the shear modulus by $\Sigma_\mu > 1$, the rotational distortion will become less and less prominent. Fig. 4 illustrates this effect by showing that the larger the value of Σ_μ, the larger the inverse flattening in each particular layer. The curves were again computed with $\nu=1$ for the pseudo-static case, that is for a forcing period of 330 hr.

4.6 Rotational distortion of realistic spherical Earth models

The use of incompressible Earth models has some mathematical convenience, it has the advantage of simulating closely Clairaut's (1743) original hypothesis, and it is compatible with a number of papers concerned with the related problem of computing Love numbers and rotational deformation (e.g. Lefftz 1991; Lefftz & Legros 1992, and references cited therein). However, using an algorithm dealing with more realistic compressible Earth models seems preferable to us. Indeed, once the necessary coding has been performed, it is as easy to cope with realistic models as with incompressible models. To this end we adapted our free oscillation program for the computation of Love numbers and rotational distortion, and named it RotationalLoad. It should be mentioned that all the computations reported in this paper used 96-bit arithmetic.

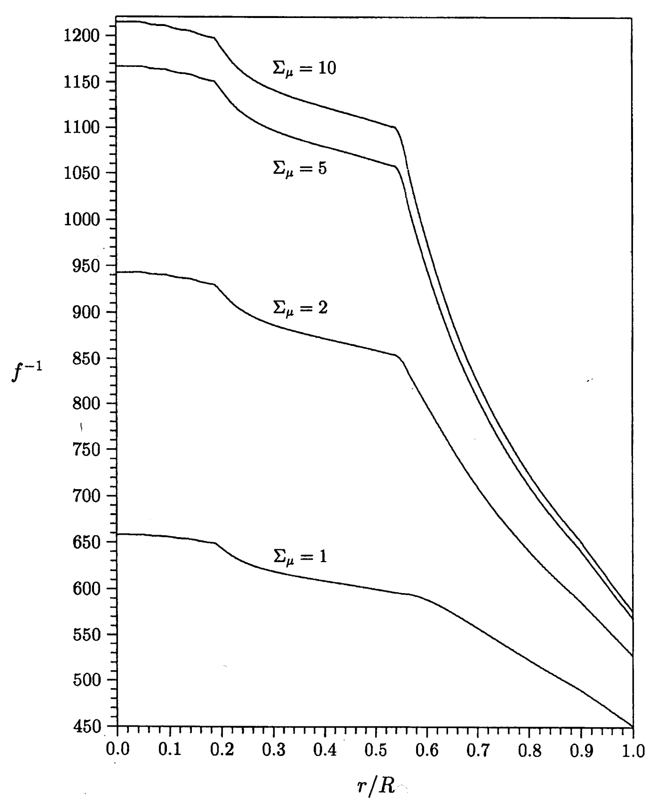

Figure 4. Inverse flattenings f^{-1} of equipotential surfaces for models of increasing shear moduli (i.e. for $\Sigma_\mu=1$, 2, 5 and 10.) The basic density model used for computing these curves is the same as that of Fig. 3, namely a model obtained by representing the density structure of PREM with 50 uniform incompressible layers of equal thickness.

In the solid parts of the Earth, we integrate the sixth-order differential system (101). In the fluid parts, we put $\mu_o = 0$. Then (101) reduces to a fourth-order system:

$$\omega^2 U + \frac{dP}{dr} + g_o X - \frac{d(g_o U)}{dr} + \frac{1}{\rho_o}\frac{d(\lambda_o X)}{dr} = \frac{dZ}{dr}, \tag{136a}$$

$$\omega^2 V + \frac{P}{r} - \frac{g_o U}{r} + \frac{\lambda_o X}{\rho_o r} = \frac{Z}{r}. \tag{136b}$$

The Poisson equation (101c,d) remains, of course, the same.

Using, as before, a forcing period of 330 hr and $\nu = 1$, we integrated the deformation equations for models in which the density $\rho_o(r)$ and the compressional-wave velocity $\alpha(r)$ are those of PREM, and in which the shear-wave velocity $\beta(r)$ was multiplied by 1 ($\Sigma_\mu = 1.00$), by 0.5 ($\Sigma_\mu = 0.25$), by 0.1 ($\Sigma_\mu = 0.01$) and by 0.01 ($\Sigma_\mu = 0.0001$). The results for the inverse flattening f^{-1} are plotted in Fig. 5. The curve for $\Sigma_\mu = 0.0001$ coincides within the differences of the approximation involved with the curve denoted *hydrostatic*, obtained by means of the CLAIRAUT3 program. These differences are hardly discernible at the scale used for the plot. The upper dashed curve, denoted *elastic incompressible*, corresponds to the curve denoted *elastic* in Fig. 3. Comparing it with the solid curve directly below, denoted $\Sigma_\mu = 1.00$, gives an idea of the approximation involved when using incompressible layers. Thus, in the latter case, the reciprocal central flattening is 658.37 and the reciprocal surface flattening is 451.54. The corresponding values in the former case are respectively 628.26 and 444.75. The curve for $\Sigma_\mu = 0.01$ takes on values of f^{-1} which are smaller than for the curve $\Sigma_\mu = 0.0001$ in the outer core. This minimum behaviour is similar to that already found for incompressible layers, and an explanation must be sought along similar lines as before. Notice that even for periods as long as 330 hr, some dynamic effects should be present, which may be ascribed to the existence of core modes. Indeed, Denis *et al.* (1997) have shown that in PREM, gravity modes with an oscillating behaviour exist for $1221.5 < r < 1695$ km and $2319 < r < 3147$ km. The lowest quadrupolar eigenperiods of these core modes, in the non-rotating limit, are 85.6, 115.3, 234.4, 254.9 and 388.3 hr. Thus, the value 330 hr adopted for the calculations reported here is rather far from any resonance period, and the corresponding f^{-1} values may loosely be called 'static' values. Moreover, it may be worthwhile noting that for a forcing period of 330 hr and $\Sigma_\mu = 0.01$, the values of f^{-1} in the mantle differ again only slightly from those calculated in the incompressible case. This can easily be seen by comparing the relevant curve in Fig. 5 with

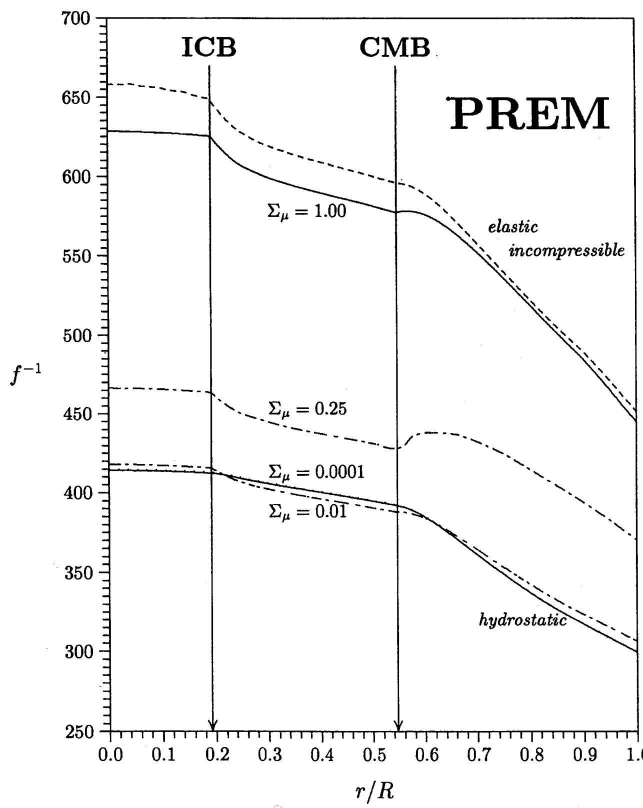

Figure 5. Inverse flattenings f^{-1} of equipotential surfaces for models with the same density and *P*-wave distributions as PREM, and with shear moduli which are equal to the shear modulus of PREM multiplied by the scale factors $\Sigma_\mu = 1.00$, 0.25, 0.01 and 0.0001. The curve denoted 'elastic incompressible' is the curve named 'elastic' in Fig. 3, and the curve denoted 'hydrostatic' is the exact hydrostatic solution obtained by means of our code CLAIRAUT3.

the corresponding curve in Fig. 3. Thus, the central and surface values of f^{-1} are 418.12 and 306.52 respectively for the compressible PREM, and 421.36 and 307.01 respectively for the incompressible PREM.

We shall investigate in more detail the effect of core resonances in another paper. Here, we merely quote results of two kinds. First, it may be instructive to see what happens for a model in which the core structure is very different from that of PREM. Thus, we computed reciprocal flattening curves for our Earth model CGGM (Denis, Denis-Karafistan & Delire 1991; Denis *et al.* 1997; Denis & Rogister 1994), assuming $\nu=1$ and either $\Sigma_\mu=0.01$ or $\Sigma_\mu=0.0001$. We considered here a forcing period of 100 instead of 330 hr. The results are illustrated in Fig. 6 where, for comparison purposes, we have also plotted the corresponding curves for PREM. We at once notice that in CGGM the reciprocal flattening is larger for $\Sigma_\mu=0.0001$ than for $\Sigma_\mu=0.01$ in both the inner core and the outer core, whereas in PREM this happens only in the outer core. For CGGM, core modes with an oscillatory time behaviour exist in the region situated between the ICB ($r_{\mathrm{ICB}}=1221.5$ km) and a radius of about 2478 km (Denis *et al.* 1997). The values of the smallest eigenperiods pertaining to the harmonic degree $\ell=2$ are 84.2, 113.6, 143.3, 173.0 and 202.8 hr.

Second, we should like to point out that for another Earth model, with a core structure different both from PREM and CGGM, namely the model PREMM (Denis & Rogister 1994; Denis *et al.* 1997), we ran into severe instability problems for forcing periods even as small as 100 hr. These problems disappear if we choose $\Sigma_\mu=0$ or Σ_μ close enough to zero. As far as our understanding of this problem goes at present, the unstable numerical behaviour seems to be related to the fact that PREMM has an entirely unstable layering, and thus no stable gravity core modes, only unstable convective modes. Further careful examination is necessary, but we note that the paper by Pekeris & Accad (1972) on core dynamics provides some insight into the problems to be dealt with. In fact, we are facing here what has become known as Longman's static core paradox, which will be discussed below. It occurs to us that for realistic Earth models no static solution is possible. This means that for a sufficiently long forcing period we are liable to run into numerical difficulties if we try to compute a static deformation of any compressible Earth model, unless the latter possesses a liquid core which is in strict neutral equilibrium.

Before discussing the specific problem of static deformation, we should like to show an actual numerical example of a dynamic deformation caused by a periodic change of LOD with an amplitude of 1 ms at a diurnal rate, using PREM as a realistic Earth model. The associated Love numbers are of course identical to the diurnal tidal Love numbers pertaining to the outer surface, namely $h=0.6121$ for the radial displacement, $l=0.0860$ for the tangential displacement (Shida number), $k=0.3029$ for the change of the gravitational potential, $\gamma=0.6908$ for the clinometric factor, $\delta=1.1577$ for the gravimetric factor, and $\Lambda=1.2169$ for the astrometric factor. Fig. 7 shows that the flattening curves as defined by (134) and (135) for the equipotential and the material flattenings, respectively, are very different, even including the sign. In absolute value, the material strata are less affected by the

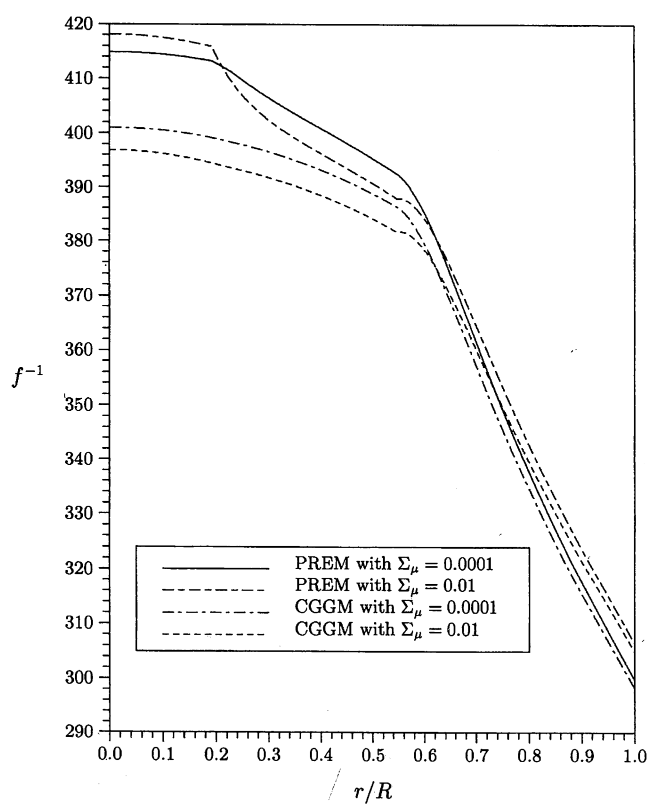

Figure 6. Inverse flattenings f^{-1} of equipotential surfaces for models with the same density and P-wave distributions as PREM and CGGM, and with shear moduli which are equal to the shear modulus of PREM or CGGM multiplied by the scale factors $\Sigma_\mu=0.01$ and 0.0001.

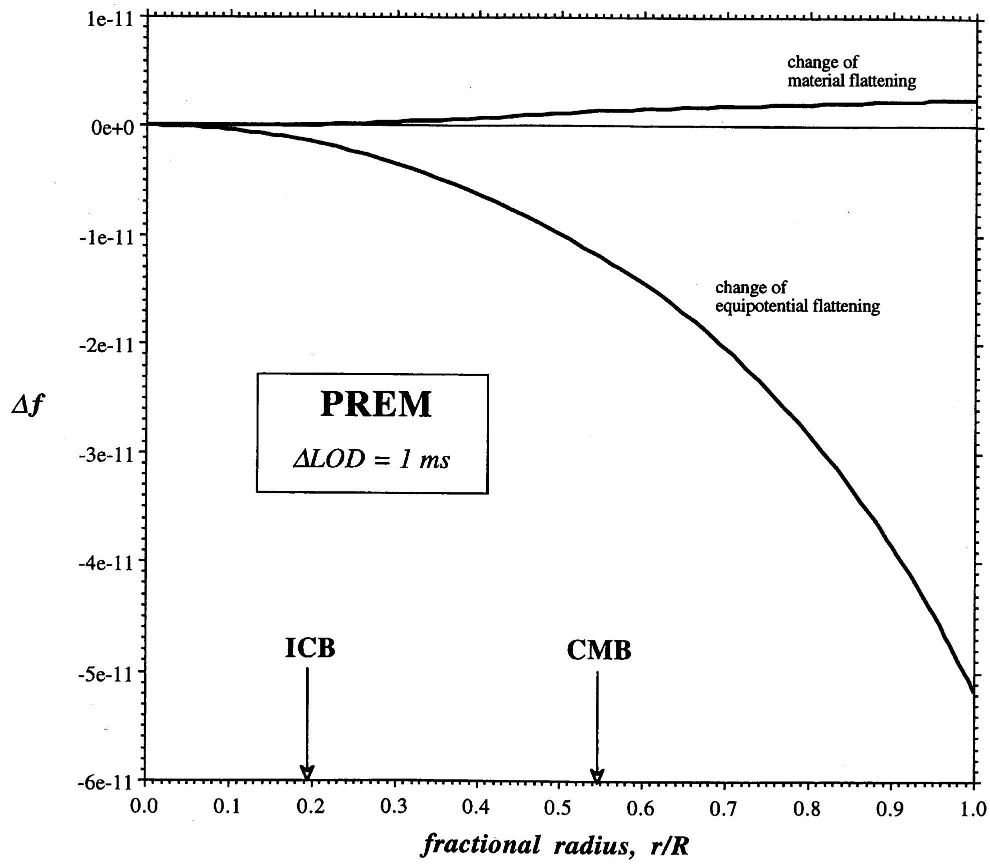

Figure 7. Change in the flattening of the isopycnic (upper curve) and the equipotential (lower curve) strata of PREM caused by a change of LOD of 1 ms at a diurnal frequency.

change of the centrifugal potential than the equipotential strata. Fig. 8 shows the corresponding changes in the radial factors of the radial and transverse displacements, and Fig. 9 the corresponding changes in the radial factors of normal traction and normal shear stress as defined in (112) and (113).

5 STATIC DEFORMATIONS AND THE LIQUID CORE PARADOX

5.1 Longman's original problem

More than three decades ago, Longman (1962, 1963) attempted to construct a Green's function for determining the global deformation of a spherical Earth model caused by static mass loads corresponding to an arbitrary spherical harmonic. With minor changes, this amounts to the same problem as computing static tidal Love numbers or rotational Love numbers. Longman found that, unless the density structure of the liquid core obeys the Adams–Williamson equation, no solution is possible. The mathematical proof is easy.

Indeed, consider eqs (136a,b) with $\omega^2=0$, namely

$$\frac{d\varpi}{dr}+g_o X-\frac{d(g_o U)}{dr}+\frac{1}{\rho_o}\frac{d(\lambda_o X)}{dr}=0\,, \tag{137a}$$

$$\varpi-g_o U+\frac{\lambda_o X}{\rho_o}=0\,, \tag{137b}$$

where we have written $\varpi=P-Z$ for the radial part of the net perturbation of the gravity potential. Eq. (137a) may be written as

$$\frac{d}{dr}\,[\rho_o(\varpi-g_o U)+\lambda_o X]+g_o\rho_o X-(\varpi-g_o U)\,\frac{d\rho_o}{dr}=0\,. \tag{138}$$

Taking account of eq. (137b), the terms between brackets in (138) amount to zero, thus leaving

$$g_o\rho_o X-(\varpi-g_o U)\,\frac{d\rho_o}{dr}=0\,. \tag{139}$$

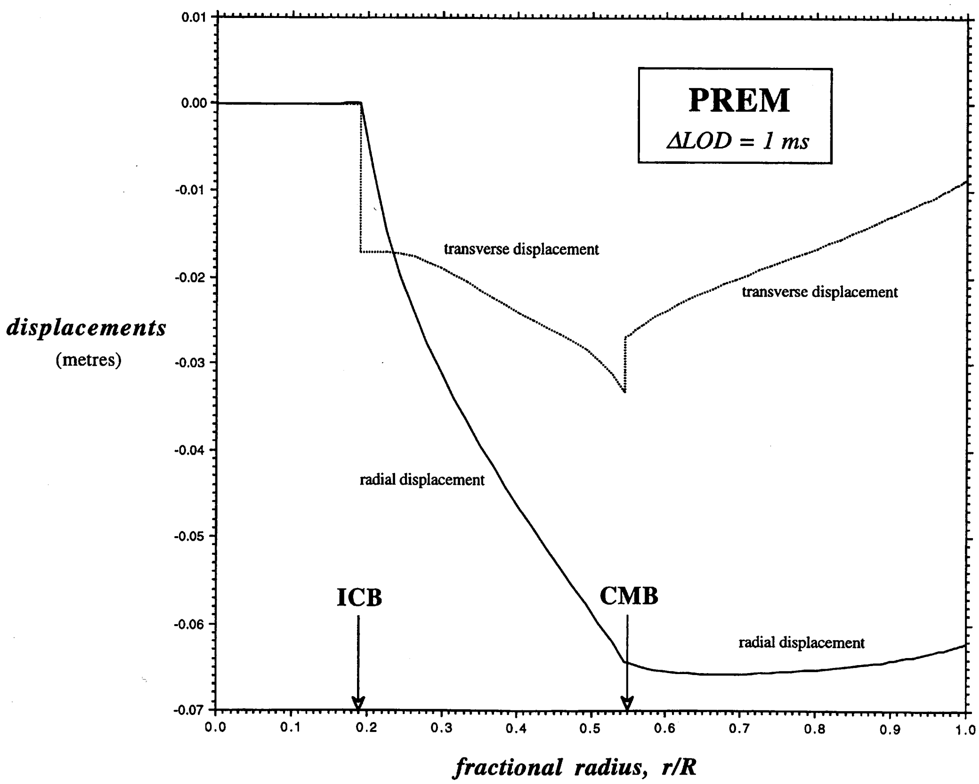

Figure 8. Change in the radial factors of the radial and transverse spheroidal displacement components of PREM caused by a change of LOD of 1 ms at a diurnal frequency.

Further consideration of eq. (137b) shows that (138) may also be written essentially as

$$N_o^2(r)X(r)=0\,, \tag{140}$$

where N_o^2 is the square of the local Brunt–Väissälä frequency, i.e.

$$N_o^2(r)=-\left(\frac{1}{\rho_o}\frac{d\rho_o}{dr}+\frac{\rho_o g_o}{\lambda_o}\right)g_o\,. \tag{141}$$

Thus, for a static situation to hold in a fluid layer which is deformed by a harmonic potential field, the deformation must either be *incompressive* (i.e. divergence-free: $X=0$), or the local Brunt–Väissälä frequency must vanish everywhere in the fluid layer ($N=0$), or both conditions may be fulfilled. By Schwarzschild's (1906) criterion, $N^2(r)=0$ denotes a marginally stable layer, $N^2(r)>0$ a gravitationally stable layer, and $N^2(r)<0$ a convectively unstable layer. In a geophysical context, $N^2(r)=0$ is referred to as the Adams–Williamson equation. In the latter case, the medium is mineralogically homogeneous and the layering is isentropic. *A priori,* there is no reason to assume that the Adams–Williamson condition is strictly fulfilled everywhere in the outer core. Thus, by eq. (140), a static deformation field has to be solenoidal within the liquid core. Under these circumstances, however, eqs (137b) and (101d) require that the static displacement field of mass points, $U(r)$ and $V(r)$, be completely determined by the displacement of the level surfaces, in particular

$$U=\frac{\varpi}{g_o}\,. \tag{142}$$

The dependent variable ϖ obeys in the fluid core the following second-order homogeneous ordinary differential equation, derived from Poisson's equation (101c) and the fact that Z fulfills Laplace's equation:

$$\frac{d^2\varpi}{dr^2}+\frac{2}{r}\frac{d\varpi}{dr}-\left[\frac{\ell(\ell+1)}{r^2}+\frac{4\pi G}{g_o}\frac{d\rho_o}{dr}\right]\varpi=0\,. \tag{143}$$

Integration of the latter leaves us with two arbitrary constants in the fluid core, to be determined by the appropriate boundary conditions. All other dependent variables of the problem, U, V, etc., are determined in the fluid core in terms of these two constants.

At the time Longman was dealing with this problem of static deformation, the currently available Earth models were models with a liquid core throughout, from the centre to the CMB. Then, for the displacement to remain finite at the centre, one of the two integration constants must be zero, leaving only one undetermined constant in the core. A second undetermined constant arises from the fact that the CMB should be modelled as a slip boundary, thus allowing an arbitrary jump in V. However, these two arbitrary

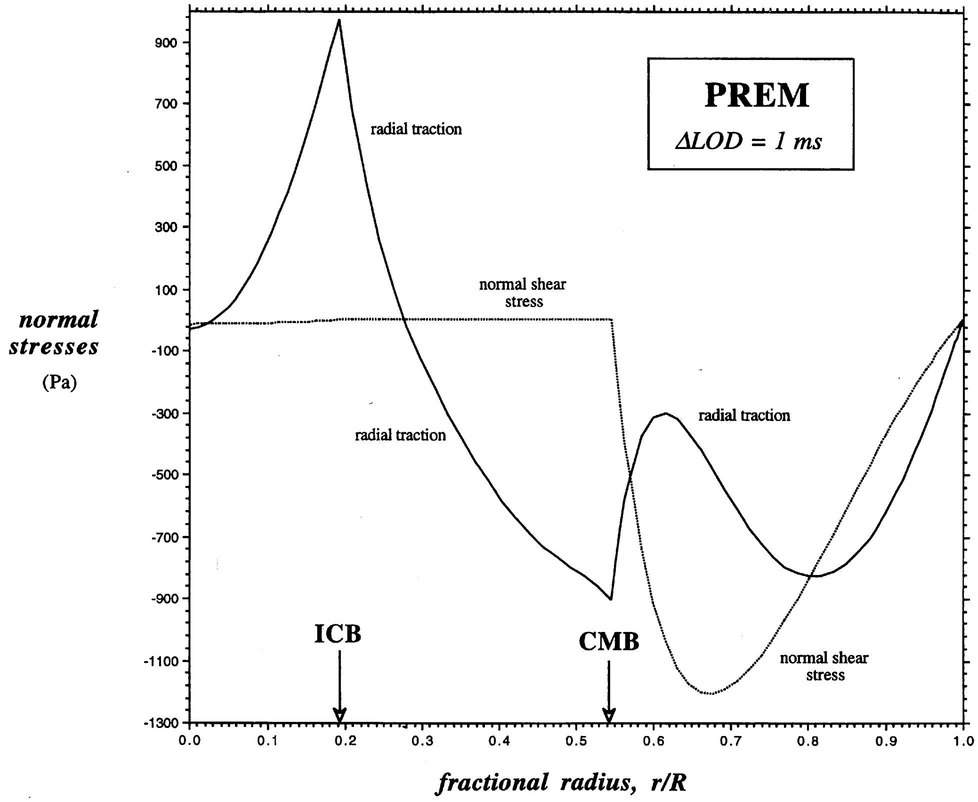

Figure 9. Change in the radial factors of the radial and transverse spheroidal normal stress components of PREM caused by a change of LOD of 1 ms at a diurnal frequency.

constants are in general not sufficient to fulfil the three boundary conditions at the outer surface, namely $R(a)=S(a)=Q(a)=0$. The obvious conclusion therefore is that *an Earth model consisting of a solid mantle and a fluid core cannot, in general, be in a state of static equilibrium.* This conclusion is hardly altered if we consider a solid inner core. Indeed, the solid inner core provides three arbitrary integration constants, the fluid outer core two, and the solid mantle six, yielding a total of 11 arbitrary constants to fulfil a total of 13 constraints (five continuity conditions at the ICB, five continuity conditions at the CMB and three boundary conditions at the outer surface). Again, a solution is thus impossible in general.

Having given this mathematical proof, Longman (1963) pointed out, quite correctly, that a static solenoidal displacement field in a non-neutrally stratified core for a given spherical harmonic is incompatible with a static displacement field in the mantle if the CMB is assumed to be a simple interface. From this he inferred, quite incorrectly, however, that the Earth's liquid core ought actually to be neutrally stratified (see also Longman 1966, 1975).

In his pioneering study on static tidal deformations of a realistic Earth model, Takeuchi (1950) should actually have come across the same problem. However, from his implicit statement (p. 678) that the Adams–Williamson condition is automatically fulfilled in a liquid, we may conclude that he probably remained unaware of the problem at that time, despite the fact that Love (1911, p. 50 and p. 56) had already been fully aware of it, and had suggested that a static tidal solution would be impossible if there was a liquid layer within the Earth. Takeuchi (1950, p. 679) discussed briefly the density structure in the core, and finally adopted the density distribution assumed by Bullen (1936) in his original paper on the Earth's structure, namely a quadratic Roche law. He then used this density law, the hydrostatic equilibrium equation and the Adams–Williamson equation—the latter in agreement with his erroneous belief that it should *necessarily* hold in a fluid—to compute values of the bulk modulus in the liquid core. Takeuchi's paper largely contributed in making the idea that a static response of the Earth to tidal or surface mass loads necessarily exists become an unquestioned dogma. This belief might have been strengthened by the fact that static tidal solutions do indeed exist for entirely solid or entirely fluid models. Thus, analytical solutions for uniform, incompressible, self-gravitating spheres had already been derived in the nineteenth century by Lord Kelvin (Thomson 1863), and played a major role in the early history of theoretical geophysics.

As noted above, having shown that a static solution was impossible if the Adams–Williamson condition did not hold in the liquid core, Longman (1963, 1966) concluded that the Earth's core must be neutrally stratified. However, it soon became obvious and generally accepted that this conclusion makes no physical sense. Indeed, the structure of the core is a result of composition and complex thermodynamic phenomena, not of static deformation, and is unlikely to obey strictly the Adams–Williamson equation (Jeffreys & Vicente 1966; Kennedy & Higgins 1973; Le Mouël, Poirier & Assoumani 1994; Labrosse, Poirier & Le Mouël 1997). Of course, this understanding does not settle the problem encountered by Longman. The latter has become known as 'Longman's paradox' (Chinnery 1975), as the 'static core paradox' (Dahlen & Fels 1978) or as the 'Jeffreys–Vicente dilemma' (Smylie & Mansinha 1971a). It is noteworthy that the problem concerns the response of the Earth to a time-invariant stimulus acting globally according to a spherical surface harmonic of a fixed degree. The question is not originally one concerning the permanent displacement field produced by a seismic point source, or by an earthquake dislocation of limited extent.

Jeffreys & Vicente (1966) checked that the different forms of the elastic equations used by various authors (Love 1911; Jeffreys 1929; Takeuchi 1950; Jeffreys & Vicente 1957; Pekeris & Jarosch 1958; Alterman *et al.* 1959; Molodensky 1961; Longman 1962, 1963; Alsop & Kuo 1964) were equivalent, and gave some thought to Longman's problem. They remained vague, however, as to its solution, suggesting merely that a solution may be impossible, or that a definite relation may exist between the external potential and the normal stress at the boundary. They pointed out, moreover, that tidal Love numbers computed with a static procedure by Takeuchi (1950) disagreed by several per cent with the numbers obtained by extrapolating to an infinite tidal period results computed with a dynamic procedure. This problem was later reconsidered in detail by Pekeris & Accad (1972), and followed up by Denis (1974, 1979). Their asymptotic theory of long-period bodily tides provides interesting insight into rotation-free core dynamics, but fails to give a definite explanation of Longman's paradox.

5.2 On the existence of static solutions

What Longman demonstrated mathematically but was not ready to accept on erroneous physical beliefs, is that there can be no static displacement field in a solid model comprising a fluid layer in which the density profile is not given by the Adams–Williamson equation. Thus, the concept of *static* Love numbers is not meaningful for realistic Earth models, and should be banished from the geophysical vocabulary. It can only lead to non-scientific debates and incorrect results, of which the lasting debate about the period of the Slichter mode provides a striking instance (*cf.* Smylie 1992; Crossley, Rochester & Peng 1992; Smylie & Jiang 1993; Hinderer & Crossley 1993; Denis *et al.* 1997). It is, therefore, important to realize that *the controversy about static deformation only arises because a static solution is assumed to exist necessarily,* and to represent a limiting simple case of a long-periodic deformation. In fact, there seems to be no essential need to resort to static deformation.

The reason for the non-existence of a static solution in realistic Earth models is easy enough to grasp. Let us assume that we constrain the core material to be displaced by a load potential in such a way as to fulfil eq. (142). The core is then in a state of static equilibrium and the core boundary is a level surface. Because we assume this boundary to be a simple interface, the mantle material on the other side must espouse exactly the same shape. We are thus led to investigate an elastostatic problem in the mantle, with prescribed boundary conditions at the CMB and at the outer surface. To prove the existence of solutions to the elastostatic equations with prescribed boundary conditions, even without considering perturbations in gravity, is generally a very difficult problem, and is far from being completely solved yet (e.g. Fichera 1972; Gurtin 1972, pp. 102–110; Solomon 1968, pp. 111–113; Wang & Truesdell 1973, pp. 470–543). However, in the case investigated here, it should be clear that no solution exists, because of the particular boundary conditions considered. Cohesive forces act in solids but not in fluids. Thus, there exists on the mantle side of the CMB, along with the same gravity potential as on the core side, an elastic strain potential which is not a minimum under the prescribed conditions and, therefore, leads to motion in the mantle.

On the other hand, if we were to constrain the mantle to be static, the equilibrium shape of its lower boundary would not match a level surface in the core. Therefore, if the core is either stably or unstably stratified, buoyancy will act as a net restoring force, and motion will set in. If the core is neutrally stratified, no net restoring force will occur in the core. Therefore, equilibrium may be static in both the mantle and the core. In other words: a static solution is generally impossible because the shape of the CMB on the core side is only determined by the gravity potential, whereas its shape on the mantle side is determined by the same gravity potential plus a non-vanishing elastic strain potential. Such a shape mismatch, which gives rise to unbalanced restoring forces, does not occur if the model is entirely solid or entirely fluid.

We note that for Earth models made up of incompressible uniform layers, there is no problem in obtaining a static solution, even if some layers are liquid and others are solid (*cf.* Section 4.5). This case in particular has been studied carefully by Rogister (1994, 1995). The reason for the existence of a solution in this case is that in each fluid uniform incompressible layer, $N^2(r)=0$ because $d\rho_o/dr=0$ and $\lambda_o=\infty$. In a neutrally stratified layer, no work has to be performed against buoyancy forces to maintain equilibrium, and thus the displacement field is essentially arbitrary. We suggest choosing for the static particle displacements the limit values we obtain if we extrapolate the dynamic results to zero frequency.

As far as the incompressible models are concerned, one should be aware of an interesting statement made in a footnote of the excellent book by Wang & Truesdell (1973, p. 237). According to these authors, there is at present no theorem giving a precise status to the solutions obtained for incompressible bodies which would allow us to consider them as limits of some appropriately selected family of solutions for certain corresponding compressible bodies. For any compressible material the entire stress is determined by the deformation, but in the limit of incompressibility we must presume that an additional arbitrary hydrostatic stress suddenly appears. This point is by no means trivial. Also, there is no unambiguous definition of the particular incompressible material to which a given compressible one should 'reduce' in such a passage to the limit. Therefore, we believe that many of the geodynamical results derived under the so-called 'simplifying' assumptions of incompressibility and staticity should be considered with some caution and, whenever possible, such mathematically convenient assumptions should be replaced by more physical ones. Before providing further details on the controversy about static displacements, we should like to recall another interesting situation where no static equilibrium can exist unless a very unlikely physical constraint is fulfilled. This situation arises for a rotating self-gravitating fluid mass in which there occurs internal heat generation at a rate ε. Von Zeipel's theorem (Zeipel 1924a,b; Eddington 1926, 1929; Jardetzky 1929, 1958; Wasiutyński 1946; Verhoogen 1948; Vogt 1957) states that under these circumstances relative hydrostatic equilibrium can exist only if ε is proportional to the quantity $1-\Omega^2(2\pi G\rho)^{-1}$. Here, Ω denotes the constant spin rate, and ρ is the density function. Von Zeipel's theorem has some implications about the possibility of the existence of *static* figures of equilibrium. It demonstrates that, contrary to what common sense would suggest, the concept of a static cosmic body is not trivial. The physical principles underlying the theorem are very simple. We start with the assumption that hydrostatic equilibrium exists in a compositionally homogeneous body, and this assumption requires the temperature to be constant on a level surface. This requirement, embodied in the celebrated Poincaré–Wavre theorem (Poincaré 1893; Wavre 1932; Jardetzky 1958; Tassoul 1978; Denis 1989; Moritz 1990), leads to a specific relation between heat production rate and spin rate, which has little chance of being fulfilled by Nature. Although the physical context here is rather different from that involved in the static core paradox, there is a striking similarity. Indeed, in the latter case equilibrium in both the mantle and the core can be achieved only if the material surfaces, which in the unperturbed body coincide with given equipotential surfaces, remain equipotential surfaces in the deformed body. The existence of an elastic shear strain potential in the mantle makes this impossible, unless the very unlikely situation of a neutral stratification in the core occurs.

5.3 On a subtle change in Longman's original formulation

Pekeris & Accad (1972), Dahlen (1974), Wunsch (1974, 1975), Dahlen & Fels (1978) and others considered the original problem of Longman, which is a problem of statics to be dealt with by means of purely statical methods, as a *limiting case* of a dynamic problem. They claimed that the difficulties encountered for static deformation in a non-neutrally stratified liquid core were due to the existence of a gravity mode spectrum which becomes dense at zero frequency. This inference, which lastingly obscured the exact nature of Longman's problem, is essentially incorrect, as we have shown above. The fact that static solutions exist separately for entirely fluid or entirely solid spheres clearly indicates that there is no reason to assume that the core rather than the mantle is causing the difficulties, or vice versa. If we were to keep the core static by some device or other, motion would occur in the mantle as a consequence of unbalanced elastogravitational restoring forces associated with an elastic mode spectrum, which becomes dense at infinite frequency. If we were to keep the mantle static, motion would occur in the core as a consequence of unbalanced restoring buoyancy forces associated with a gravity and/or a convective mode spectrum, which both become dense at zero frequency. A mode with zero frequency is in fact of no concern because it does not give rise to any restoring force; thus, in the case of neutral equilibrium, the gravity/convective mode spectra are degenerate: all the modes are associated with a null frequency.

Press (1965) observed a residual strain signal of global extent after the 1964 Alaskan earthquake. Coming soon after Munk & MacDonald's (1960) outstanding book on geophysical aspects of the Earth's rotation, as well as dramatic progress in space geodesy (e.g. Anderson & Cazenave 1986), which had made theoretical studies on the variations in terrestrial rotation a fashionable topic, the observations of Press stimulated research on the excitation of the Chandler wobble by seismic displacement fields, and seemed to prove the existence of a static displacement field.

In particular, much work was devoted in the late sixties and early seventies to constructing such static displacement fields in order to evaluate changes in the Earth's inertia tensor (e.g. Mansinha & Smylie 1967; Smylie & Mansinha 1971a; Rice & Chinnery 1972; Dahlen 1971b, 1973). Of course, all these studies encountered the difficulties involved in Longman's formulation of the static deformation problem. Because the existence of a global static solution was accepted as a fact and hardly ever questioned seriously, contradictory claims appeared concerning the nature of the boundary conditions to be imposed at the CMB, and the determinacy or indeterminacy of the displacements in the core (Dahlen 1971a; Smylie & Mansinha 1971b).

Many authors contributed to the controversy with physical arguments which essentially express a general misunderstanding of the nature of the problem (Israel, Ben-Menahem & Singh 1973; Denis 1974, 1979; Saito 1974; Smylie 1974; Chinnery 1975; Crossley & Gubbins 1975; Longman 1975; Lanzano 1982). The most common belief expressed was that the radial displacement U is discontinuous at the CMB in a static displacement, and this hypothesis still flaws an undetermined, but certainly too large, number of computer programs. Thus, in particular, Smylie & Mansinha (1971a,b) assumed U to be discontinuous at the CMB, and tried to justify this assumption by claiming that particle displacements are indeterminate because particles in the core are indistinguishable from each other. (We wish to point out that such a principle indeed holds in quantum physics, but not in classical physics and certainly not in macrophysics, where it would ruin the very basic concepts of continuum mechanics.) According to these authors, the mantle can 'bump' into the core like a boat on a lake, the core liquid moving around the mantle bump like the water of the lake moves around the boat. This kind of physical argument, which seems quite appealing at first sight, is typical of many of the erroneous arguments used by the authors involved in the liquid core debate. In fact, the 'boat-on-a-lake' analogy is quite misleading, because the boat problem is a local problem, whereas the static core problem is a global one. Indeed, buoyancy produced by putting a boat on the surface of a lake, which amounts to the weight of the displaced volume of water, exactly balances the weight of the boat, and there is no net restoring force. The displaced volume of water rises the water level around the boat by a finite amount, although this rise will be very tiny, the larger the lake the tinier the rise. Thus, isostatic equilibrium can be achieved in this situation because the surface of the lake around the boat is a free surface. The core problem is quite different, because the lower boundary of the mantle acts like an elastic membrane, not like a stress-free surface. Buoyancy forces produced by a mantle bump which intersects different level surfaces in the core cannot be balanced because the CMB is a closed surface.

Much confusion arose when Dahlen (1974) attempted to clarify the situation by considering static deformation as a *motion*. He claimed that the gravity mode spectrum of the liquid core was responsible for the difficulties encountered, which he believed to be of a mathematical nature. Moreover, he stated that the consideration of a Eulerian, rather than a Lagrangian, formulation solved the problem. To us, this seems to be a somewhat unintelligible way of getting around the difficulties. First, the existence of a velocity field is a primordial pre-requisite for a Eulerian formulation of continuum mechanics. In a static deformation, there is no motion and thus no velocity field, only Lagrangian displacements. Dahlen confuses a static deformation with a stationary motion. Second, in the case of motion, it can be demonstrated that the Eulerian and Lagrangian descriptions are mechanically equivalent. Thus, any problem which can be solved in one representation can, in principle, also be solved in the other. These criticisms, which were already formulated by one of us immediately after Dahlen's paper appeared (Denis 1974), should nevertheless be mitigated by stressing that the paper is all in all an excellent one, with deep insights into various problems and fully justified criticisms of work on the static core paradox by other authors. Unfortunately, Dahlen's confusing concepts, in particular the idea that the liquid core was responsible for the difficulties encountered, received support from Wunsch (1974, 1975) and, with a few amendments, evolved into some kind of a firm belief.

Longman (1962, 1963) had been interested in the problem of computing the Earth's yielding due to a static surface mass load field corresponding to a fixed spherical harmonic. With only minor changes, this is the same problem as computing a static tidal response or a static rotational deformation. Most subsequent authors, however, had in mind the problem of computing the static displacement field resulting from a seismic source function with a behaviour in time involving one or several step functions. The aim was mainly to assess whether major earthquakes are capable of exciting the Chandler wobble to a significant amplitude. As such, the problem maintains its interest today. Although the equations describing the static displacement field are the same in both cases, the point of view undergoes a subtle change when considering an internal earthquake source instead of an external load. This change may at least partially explain the considerable amount of confusion associated with Longman's paradox.

Indeed, when dealing with a deformation forced by a spherical harmonic load of a particular degree ℓ, we assume in the linear theory adopted here that the material elements initially situated on a spherical level surface will get displaced in such a way that the angular dependence of the resulting displacement field corresponds only to a spherical surface harmonic of the same degree as the load. Thus, the global shape of any layer in the deformed model is a superposition of a spherical harmonic of degree 0 and a spherical harmonic of degree ℓ. This formulation reduces to a problem of hydrostatics in the liquid core, and to a problem of elastostatics in both the mantle and the solid inner core.

When dealing with an earthquake source, however, we generally have in mind a point source or a source of limited spatial extension. The displacement field produced by such a source can be described in time and space as a sum of excitation functions involving source characteristics and normal modes associated with spherical harmonics of various degrees and orders (Gilbert 1970). There is no basic difficulty in computing any desired normal mode for an Earth model containing a liquid core. Thus all necessary excitation functions can be determined in principle. The static response is the residual displacement field left by this sum of elementary excitation functions when all transients have died away. This is the situation discussed by Dahlen & Fels (1978) in a scholarly paper which they consider solves definitively the static core problem, but which in fact addresses a problem quite different from Longman's.

Our main conclusion is that the concept of static Love numbers is meaningless for solid planets possessing a liquid core, and should be banished from future work on Earth rotation and geodynamics.

6 EFFECT OF A SMALL DIFFERENTIAL ROTATION

It is worth mentioning that the theory of figures discussed in Section 3 can be extended to include differential rotation of a particular kind. This has already been indicated by Trubitsyn *et al.* (1976), Zharkov & Trubitsyn (1978) and Denis & Goerlich (1985a,b).

When trying to generalize the CLL theory of figures, it is essential to note that the latter relies entirely on the concept of an equipotential surface. Therefore, an obvious request is that we should be able to define such a level surface $U(x, y, z) = \text{constant}$, where U denotes here the effective gravity potential, the sum of the gravitational potential V and the centrifugal potential Z. It is now possible to define a centrifugal potential field only under very restrictive conditions. The latter are easily established. Indeed, consider the effective gravity components in cylindrical coordinates h, z, λ, namely

$$g_h = -\frac{\partial V}{\partial h} + \Omega^2 h, \qquad g_z = -\frac{\partial V}{\partial z}, \qquad g_\lambda = 0. \tag{144}$$

These equations show that a rotational potential Z, and therefore a level surface $U = V + Z = \text{constant}$, can exist *if and only if* the spin rate Ω is independent of z. The latter amounts to saying that a rotational potential can be defined only if the angular velocity is constant over cylindrical surfaces centred about the axis of rotation:

$$\Omega = \Omega(h). \tag{145}$$

Instead of the usual definition of the centrifugal potential for rigid body rotation,

$$Z=-\frac{1}{2}\,\Omega^2h^2\,, \tag{146}$$

we write

$$Z=-\int_0^h \Omega^2(h')h'\,dh'\,. \tag{147}$$

Obviously, for $\Omega=$constant, the particular case of rigid body rotation, eq. (147) becomes eq. (146). Following the steps initially taken by Trubitsyn *et al.* (1976), Denis & Goerlich (1985b) generalized the CLL theory by considering, instead of a solid body rotation $\Omega=$constant, a more general differential rotation of the particular form

$$\Omega^2=\left(1+b_2h^2+b_4h^4+b_6h^6+\cdots\right)\Omega_o^2\,. \tag{148}$$

Here Ω denotes the spin rate at a distance h from the rotation axis, and Ω_o is the spin rate on the rotation axis. Trubitsyn *et al.* (1976), and again Zharkov & Trubitsyn (1978, p. 289) assume that the coefficients b_2, b_4, b_6, ... must be smaller than or equal to the dynamic constant $m=\Omega^2R^3/GM$. Goerlich (1985) has pointed out that the latter hypothesis is not enough, by showing analytically the CLL procedure to be consistent with a differential rotation law of the form given in (148) only if $b_{2n}=O(m^n)$. Hence, some results for the giant planets Jupiter and Saturn established by Zharkov & Trubitsyn on the basis of their assumption need not be numerically correct. Our numeric program CLAIRAUT3 contains a switch to select non-zero values for the coefficients b_2, b_4, b_6. Using this code, Denis & Goerlich (1985a) found that the effect of differential rotation is quite significant on the innermost layers of Jupiter, even if we consider a plausible value of b_2 as small as 0.02. For the Earth with its present structure, a differential rotation law of the form (148) seems impossible. Nevertheless, to get an idea of how much a small differential rotation may affect internal flattening, we made two runs for PREM with respectively $b_2=b_4=b_6=0$ and $b_2=0.001$, $b_4=b_6=0$. In this case, differential rotation reduces the flattening of each stratum by an amount which remains everywhere close to 0.035 per cent. We need not consider such small effects in a first-order theory.

Song & Richards (1996) claimed recently that they had found seismological evidence for differential rotation of the Earth's inner core. The inferred rotation rate is on the order of 1° per year faster than the daily rotation of the mantle and crust. Although very slow, such a rotation rate is not amenable to the form (148) and, strictly speaking, would render the concept of a gravity potential and level surfaces meaningless for the Earth. Thus, the CLL figure theory is unable to deal with the situation of a differentially spinning inner core.

7 FIRST-ORDER HYDROSTATIC THEORY

Until now, we have implicitly assumed that the density $\rho(s)$, or equivalently $\delta(\sigma)$, of the Earth or planet is known. Conceptually, we suppose that the density is given as a function of radius r in a non-rotating model. Then, the model is spun up in such a way that the initial spherical level surfaces are distorted *incompressively* into a spheroidal shape. Clairaut's original idea was that the different Earth strata were made up of incompressible material. In fact, we merely assume here that the distortional motion is divergence-free. According to this hypothesis, the mean radius s of the spheroidal level surface is the same as the radius r of the initial sphere. Because no compression or dilatation has occurred, the density on the distorted surface is the same as on the initial surface. It should be understood, however, that s is *not* a polar coordinate, whereas r is. Therefore, in the formulae given above, quantities such as $d\rho/ds$, $d\eta/ds$ or df/ds should not be mistaken for the radial components of the gradients of ρ, η or f, namely $d\rho/dr$, $d\eta/dr$ or df/dr. Rather, they should be interpreted as the rate of change of ρ, η or f between two different level surfaces which are infinitesimally close to the level surface referred to by the Lyapunov parameter s.

It seems noteworthy that Clairaut's differential equation (22) was not used in the early days to determine the internal flattening of the Earth, but rather as a constraint equation on the density structure. Thus, Legendre (1793) uses $\gamma=fS_0\sigma^3$ instead of f. With this new dependent variable, eq. (22) reads

$$\frac{d^2\gamma}{d\sigma^2}-\left(\frac{6}{\sigma^2}+\frac{3}{S_0\sigma}\,\frac{d\delta}{d\sigma}\right)\gamma=0\,. \tag{149}$$

For reasons which remain unclear, Legendre assumed that

$$\frac{3}{S_0\sigma}\,\frac{d\delta}{d\sigma}=-A^2 \qquad \text{(a negative constant)}\,.$$

Eq. (37) may then be written as

$$\frac{d}{d\sigma}\left(\sigma^2\,\frac{d\delta}{d\sigma}\right)=-A^2\sigma^2\delta\,. \tag{150}$$

The general solution of this second-order ODE is $\delta(\sigma)=\delta_0 \sin(A\sigma+\alpha)/(A\sigma)$, where δ_0 and α are two integration constants. Because the density ought to remain finite at the centre, $\alpha=0$ and the ensuing Legendre density law is

$$\delta(\sigma)=\delta_0 \frac{\sin A\sigma}{A\sigma}. \tag{151}$$

Laplace (1825) contributed in making Legendre's hypothesis plausible. He noted that 'it is natural to think that a liquid resists to compression the more it is already compressed'. Mathematically, this statement amounts to $dp=C\rho d\rho$, where p is pressure, ρ is density and C is a proportionality constant. On the other hand, hydrostatic equilibrium implies that

$$\frac{dp}{ds}=-\rho g=-\frac{4\pi G\rho}{s^2}\int_0^s \rho(r)r^2\,dr. \tag{152}$$

Thus, with Laplace's hypothesis we have

$$Cs^2\frac{d\rho}{ds}=-4\pi G\int_0^s \rho(r)r^2\,dr. \tag{153}$$

Taking the derivative with respect to s of eq. (153), we finally obtain

$$\frac{d}{ds}\left(s^2\frac{d\rho}{ds}\right)=-\frac{4\pi G}{C}\rho s^2. \tag{154}$$

The latter is eq. (150) written out in dimensional variables, and we are back to Legendre's result.

Roche (1848) went one step further, assuming

$$dp=\rho+C'\rho^2)\,d\rho; \tag{155}$$

(154) is then replaced with

$$\frac{d}{ds}\left[s^2(C+C'\rho)\frac{d\rho}{ds}\right]=-4\pi Gs^2\rho. \tag{156}$$

Putting $\rho_o=(3C)/(2C')$ and $\kappa=(4\pi G)/(15C)$, we can perform an immediate integration to obtain *Roche's parabolic density law*,

$$\rho=\rho_o(1-\kappa s^2). \tag{157}$$

More details concerning the application of Clairaut's theory to the early determination of the Earth's density may be found in the books of Todhunter (1873) and Bullen (1975, pp. 60–83).

We should also mention that the general CLL theory discussed in Section 3 may easily be extended to compute the density together with the flattening. For this purpose, we need to provide specific equations of state for the planetary material. Zharkov and co-workers have used the generalized CLL theory to investigate the structures of the giant planets (*cf.* Zharkov & Trubitsyn 1978). In the latter case, the geophysical CLL method competes quite well with the self-consistent field approach commonly used by astrophysicists (e.g. Hubbard, Slattery & DeVito 1975; Slattery 1977).

It is generally acknowledged that our understanding of the Earth's structure owes much to hydrostatic theory—essentially to first-order theory—but the exact amount of credit is difficult to assess. As stated above, density models for the Earth were determined during the 19th century by using Clairaut's theory in a straightforward way, and thus were first-order hydrostatic models in the most basic sense. Dramatic progress in seismology at the beginning of the 20th century gave the matter a less trivial status, however. Indeed, the fact that by far the largest part of the Earth's bulk is made up of solid rock, with a finite resistance to shear stress, poses a conceptual problem in using hydrostatic theory. In our opinion, the latter can only partly be overcome by the more general concept of rheological behaviour.

Thus, to construct the first realistic density models of the Earth, Bullen (1936, 1940) resorted to the hydrostatic equation, but in his models the seismological factor determined by means of short-periodic seismic waves plays a central part as well. We feel that there is some kind of contradiction here: determining the density by assuming on the one hand that the Earth behaves like a liquid, and on the other hand that the Earth behaves in most of its parts like a solid. By definition, hydrostaticity means that all non-hydrostatic stresses are relaxed; seismic waves, however, experience solid material behaviour characterized by non-hydrostatic stress.

Nevertheless, the use of the hydrostatic equation can be justified to a fair extent by *assuming* that hydrostatic pressure may be identified with the quantity

$$p=-\frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3}, \tag{158}$$

where σ_{11}, σ_{22}, σ_{33} are the diagonal components of the stress tensor at each point within the Earth. The quantity p is sometimes called the *octahedral pressure.* It is essentially the latter which balances, in the Earth's interior, the weight per unit area of the rocks lying above. Thus, p will necessarily increase with depth. Now, shear resistance, which is caused by atomic and molecular cohesive forces, can be expressed in terms of differences such as $\sigma_{11}-\sigma_{22}$, $\sigma_{22}-\sigma_{33}$, $\sigma_{33}-\sigma_{11}$, and has a limiting value which is in general smaller than the value reached by octahedral stress at a depth of 30 km. Clearly, this shows that the hydrostatic equation is approximately valid at large enough depths for the purpose of determining the density structure, but does not actually provide a quantitative estimate of the approximation involved.

It is less well acknowledged that modern Earth models—such as 1066-A (Gilbert & Dziewonski 1975), PEM-A (Dziewonski, Hales & Lapwood 1975), PREM (Dziewonski & Anderson 1981) or CORE11 (Widmer 1991)—still rely very heavily on the hydrostatic assumption. This is the case not only because the original starting models for inversion were Bullen-type models, but mainly because prior to proceeding to actual inversion, the straightforward problem of computing the Earth's normal modes must be solved. In practice, the latter still relies completely on the assumption of hydrostaticity. It is difficult to imagine at present any reference Earth model which would not assume hydrostatic equilibrium, in a more or less explicit way.

Having noted the pervading character of hydrostatic theory in modelling Earth structure, another question is to establish to what extent the actual Earth is close to hydrostatic equilibrium. This question is, again, not quite as trivial as many geodynamicists seem to think.

Before the launching of artificial satellites and the spectacular progress of space geodesy in the late 1950s and early 1960s, the values determined for the geometrical flattening f, the precessional constant H and the inertia coefficient y were compatible, within observational error bounds, with their theoretical values derived by means of the hydrostatic theory of figures (Bullard 1948). This fact seemed to imply that on a global scale the Earth was indeed in hydrostatic balance. As a consequence, no major inconsistency arose from using Bullen A- or B-type models to deal with the Earth's rotational parameters (Denis *et al.* 1997). The geometric flattening f was then considered as a primary constant in geodesy and ascribed the value 1/297, and accepted as an astrogeodetic constraint on Earth models.

In the Clairaut–Radau theory, f is uniquely determined by y and m. The value of the geodynamic constant m being fixed and known accurately, all Earth models with the same value of f have the same inertia coefficient y. Reciprocally, all models with the same value of y possess the same value of f. This follows at once from the boundary condition (45) and relation (46) taken at the surface $\sigma=1$, namely

$$f^{-1}=\frac{2}{5m}\left[\left(\frac{5}{2}-\frac{15}{4}y\right)^{2}+1\right] \tag{159}$$

or, reciprocally,

$$y=\frac{2}{3}-\frac{4}{15}\sqrt{\frac{5m}{2f}-1}\,. \tag{160}$$

It is understood that we are dealing here with the values of f and y at the outer boundary $\sigma=1$. With $f^{-1}=297$ and $m=3.4498\times10^{-3}$, eq. (160) yields $y=0.33344$. It was the latter value, or a value close to it, that was used as an astrogeodetic constraint for Earth models in the pre-satellite era.

An early major result of space techniques was the accurate determination of the zonal quadrupole geopotential coefficient J_2. The latter is often called the 'dynamic shape factor', but we find this name at best misleading. Indeed, J_2 is defined here as

$$J_2=\frac{C-A}{Ma^2}\,, \tag{161}$$

where C is the moment of inertia about the polar axis, A is the moment of inertia about an equatorial axis, M is the total mass and a is the equatorial radius. Thus, J_2 is really a *kinetic*, not a dynamic, parameter. We wanted to point out this fact to save all the mysterious beauty of the celebrated theorem of Clairaut (also referred to as Clairaut's algebraic equation),

$$f=\frac{m}{2}+\frac{3J_2}{2}\,, \tag{162}$$

which relates a geometric quantity (f) to a dynamic quantity (m) and a kinetic quantity (J_2). Eq. (162) is readily obtained if we consider eq. (19) for $s=R$.

Another early breakthrough of space techniques was the demonstration that the Earth was *not* in hydrostatic equilibrium on a global scale, but that the deviations from such an equilibrium state were rather small. Accurate determinations showed that the odd zonal geopotential coefficients (in particular the 'pear shape factor' J_3) and the non-zonal harmonics of the Earth's gravity field were not zero, as they should be in the case of a hydrostatic equilibrium figure. Their actual values are of the order of 10^{-6} or smaller. Thus, the question of determining the hydrostatic equilibrium shape of the Earth, namely the shape which the Earth would take if all non-hydrostatic stresses were relaxed, is definitely a tricky one. No unambiguous answer can be given.

Nevertheless, since 1960 it has become a common belief that *the* hydrostatic flattening of the Earth is very close to 1/300. This value was first stated by Henriksen (1960), was soon afterwards corrected by O'Keefe (1960) to 1/299.8, and has since been referred to as the Earth's hydrostatic flattening. We wish to stress that $f^{-1}=299.8$ is only a *conventional value*, computed, with a small correction for second-order terms, from the first-order relation (Moritz 1990)

$$f^{-1}=\frac{2}{5m}\left[\left(\frac{5}{2}-\frac{15}{4}\frac{J_2}{H}\right)^2+1\right]. \tag{163}$$

With the modern observed values of m, J_2 and H, this yields a value of $f^{-1}=299.9$.

The important point here—a point which is commonly overlooked—is that eq. (163) is *not necessarily valid* within first-order error bounds if hydrostatic equilibrium is not achieved. It holds only in so far as

$$\bar{y}=\frac{I}{MR^2}=\frac{2}{R^5\bar{\rho}}\int_0^R \rho(s)s^4\,ds \tag{164}$$

is equal, within terms of $O(m^2)$, to

$$y^*=\frac{J_2}{H}=\frac{C}{Ma^2}. \tag{165}$$

The latter has nothing to do with hydrostatic equilibrium. It results merely from the respective definitions of J_2 and H. Only if hydrostatic equilibrium is achieved can we be sure that $\bar{y}=y^*+O(m^2)$, because then we have the relation

$$\frac{\bar{y}}{y^*}=1+\frac{2}{3}(f-H), \tag{166}$$

and the ratio $\bar{y}/y^*$ differs from unity only by a quantity of the second order of smallness. If we define the mean moment of inertia as one third of the trace of the inertia tensor, the same conclusion would hold for a *uniform* ellipsoid, whether in hydrostatic equilibrium or not. However, for a non-hydrostatic structure which is not an exact ellipsoid, such as the Earth, there is no compelling reason to assume that (166) is valid (Denis *et al.* 1997).

The most obvious way to define a hydrostatic flattening for the Earth is therefore, in our opinion, to use the very accurately observed quantities m and J_2 in Clairaut's formula (162), and to determine the inertia coefficient in a way which is fully consistent with hydrostatic theory by means of eq. (160). Proceeding in such a manner, we obtain $f_{\rm hyd}^{-1}\approx 298.6$, $y_{\rm hyd}\approx 0.3320$. Second-order effects will not change these data significantly. In this way, we have *conventionally* assumed that the hydrostatic value of J_2 is identical to the observed geoid J_2, much in the same way as the J_2 of the reference ellipsoid is conventionally put equal to the observed geoid value J_2 (Denis *et al.* 1989; Denis 1989, p. 119).

Thus, an Earth model taking account of the astrogeodetic constraint $y=0.3320$, or equivalently $f^{-1}=298.6$, yields $J_2=1.082\times10^{-3}$ and $H=3.261\times10^{-3}$. To four significant figures the observed values retained in the 1980 Geodetic Reference System (GRS1980) adopted by the IUGG (Moritz 1980) are $J_2=1.082\times10^{-3}$, $f^{-1}=298.3$ and $H=3.273\times10^{-3}$. For PREM and the other modern spherically symmetric Earth models using the constraint $y=0.3308$, we have $J_2=1.072\times10^{-3}$, $f^{-1}=299.9$ and $H=3.237\times10^{-3}$. It is quite noticeable that for $y\approx0.331$, the non-hydrostatic components of J_2, f and H amount to about 1.0×10^{-5}, 1.8×10^{-5} and 3.6×10^{-5}, respectively. On the other hand, with $y\approx0.332$, these non-hydrostatic contributions are quite significantly lower, amounting to 0.0 (conventionally) for J_2, 4×10^{-6} for f, and 1.2×10^{-5} for H. Therefore, it occurs to us that the non-hydrostatic flattening excess of about 2×10^{-5}, which since 1960 has generally been claimed to be physically significant (see, however, Goldreich & Toomre 1969), is overestimated by a factor close to 5 and is only about 4×10^{-6}.

The use of relation (163) to define the first-order hydrostatic flattening, instead of the basic hydrostatic relation (162), seems to be largely due to an unfortunate circumstance: in his book *The Earth*, which had become an incontestable reference text, Jeffreys (1959) used very confusing notations. Indeed, until he came to discuss the theory of the internal field, in particular Radau's approximation, Jeffreys systematically used a for the equatorial radius, and C for the principal moment of inertia about the rotation axis. He then suddenly used a for the mean radius (our R) and C for the mean moment of inertia (our I). For those who read Section 4.03 of Jeffreys' book carefully, not cursorily, the context shows quite clearly that the meaning of the ratio C/Ma^2 is identical to our ratio $\bar{y}=I/MR^2$ of eq. (164), *not* to our ratio $y^*=J_2/H$ of eq. (165). But when time is money, who reads a long difficult text carefully? Jeffreys (1970) added the important short sentence '*a* here being the mean radius, not the equatorial radius as in 4.022' after eq. (4) of Section 4.03 on p. 183 of the fifth edition of *The Earth*, but the harm had already been done ten years previously.

Rather than using J_2 as a primary constant, as is done in GRS1980, we may also decide to choose for example $f\approx1/298.3$ as a primary observed constant, and use the basic hydrostatic relation (162) to determine a 'hydrostatic' value of J_2. In this way, we obtain $J_2^{\rm hyd}\approx1.085\times10^{-3}$, and corresponding values for H and y of about 3.266×10^{-3} and 0.3323, respectively.

Preliminary Earth models based on the latter value of the inertia coefficient, called CGGM and PREMM, have been discussed by Denis *et al.* (1991), Denis & Rogister (1994), and Denis *et al.* (1997). An increase of y by an amount of about 0.001 seems at first sight rather insignificant, but in fact may bear important consequences for the structure of the Earth's core, as suggested by the models CGGM and PREMM. The possibility of important changes in the structure of the core, in particular a jump in density at the inner core boundary which is much smaller than the jump in PREM (597 kg m^{-3}), has been questioned by Masters (private communication, 1994) and Mitrovica (private communication, 1996). They claim that Masters & Shearer (1990) have definitely shown that certain observed free modes, which are particularly sensitive to core structure, demonstrate that the density in the Earth's core cannot be much different from that stipulated by PREM. This may be so if we stick to an inertia coefficient of 0.3308, but need not be so (and we believe *is not* so) if we impose $y \approx 0.3322$ as a gross datum for inversion. The whole work of inversion with a new astrogeodetic constraint needs to be done before the debate can be settled.

In any case, the issue of this debate should be important for geodynamical modelling and the interpretation of certain gravimetric and astrometric observations. Thus, Denis *et al.* (1997) have already drawn attention to its relevance for the frequency range where we should look for the Slichter triplet. They place this domain below the frequency of 4.428 cycles per day, which is the central Slichter frequency for PREM. The structure of the core is also important for understanding the processes at work in the geodynamo. In this respect, we think that the papers of Le Mouël *et al.* (1994) and Labrosse *et al.* (1997) are quite stimulative.

The shapes of the CMB and ICB play a critical role in the theory of the Earth's nutations (see Mathews & Shapiro 1992 for a review). An important problem which remains to be solved is the significant discrepancy of at least 6 per cent between the observed period of the free core nutation (FCN) and its theoretical value computed by Sasao, Okubo & Saito (1980) and Sasao & Wahr (1981) for Earth models based on $y = 0.3308$. The latter ranges between 460 and 466 days. The former depends somewhat on the assumptions on which the data analysis is based. An early determination, using gravity tidal data, provided 434 ± 7 sidereal days (Neuberg, Hinderer & Zürn 1987). More recent gravity tidal measurements yield values between 431.6 ± 2.9 and 435.5 ± 0.9 sidereal days, with a favoured value of 434.1 ± 0.9 sidereal days (Defraigne, Dehant & Hinderer 1994). An analysis of VLBI data leads to a somewhat smaller value of approximately 431 ± 1 sidereal days (Herring *et al.* 1991; Herring & Dong 1994; Mathews, Buffett & Shapiro 1995a,b). Simple formulae, which can be found in e.g. Neuberg, Hinderer & Zürn (1990), de Vries & Wahr (1991), Mathews *et al.* (1991a,b), Dehant *et al.* (1993) and Legros *et al.* (1993), show that a critical parameter determining the period T_{FCN} of the retrograde free core nutation is the dynamic ellipticity of the core, namely

$$e_{\mathrm{c}} = \frac{C_{\mathrm{c}} - A_{\mathrm{c}}}{A_{\mathrm{c}}}, \tag{167}$$

where the subscript c denotes values evaluated at the CMB. As previously, A stands for the moment of inertia about an equatorial axis, and C represents the moment of inertia about the polar axis.

Table 2, computed by means of our code CLAIRAUT3, shows that e_{c}^{-1} is 391.9 for PREM, but only 385.6 for CGGM and 385.7 for PREMM. We conclude, therefore, that the actual dynamic ellipticity of the core computed by hydrostatic theory is about 2 per cent larger than assumed until now. This result is important, because it brings theory and observation of the period of FCN into better agreement than before, without the need for ad hoc corrections of the inner flattening which make theory inconsistent. The FCN period may be written (Sasao *et al.* 1980; Mathews *et al.* 1991a)

$$T_{\mathrm{FCN}} = \left[\left(1 + \frac{A_{\mathrm{f}}}{A_{\mathrm{m}}} \right) (e_{\mathrm{c}} - \beta) \right]^{-1}. \tag{168}$$

A_{f} is the total equatorial moment of inertia of the fluid core, i.e. $A_{\mathrm{f}} = A_{\mathrm{c}} - A_{\mathrm{i}}$, and A_{m} is the corresponding quantity for the mantle, i.e. $A_{\mathrm{m}} = A - A_{\mathrm{c}}$. The values for PREM are $A = 8.0117 \times 10^{37}$ kg m^2, $A_{\mathrm{i}} = 5.8698 \times 10^{34}$ kg m^2, $A_{\mathrm{c}} = 9.0416 \times 10^{36}$ kg m^2, $A_{\mathrm{f}} = 8.9829 \times 10^{36}$ kg m^2, $A_{\mathrm{m}} = 7.1075 \times 10^{37}$ kg m^2. These values are close to those of the Wang (1972) model considered by Sasao *et al.* (1980) and by Sasao & Wahr (1981), namely $A = 8.013 \times 10^{37}$ kg m^2, $A_{\mathrm{f}} = 9.152 \times 10^{36}$ kg m^2. The corresponding values for CGGM (Denis *et al.* 1997) are $A = 8.0464 \times 10^{37}$ kg m^2, $A_{\mathrm{i}} = 5.5144 \times 10^{34}$ kg m^2, $A_{\mathrm{c}} = 8.9384 \times 10^{36}$ kg m^2, $A_{\mathrm{f}} = 8.8833 \times 10^{36}$ kg m^2, $A_{\mathrm{m}} = 7.1527 \times 10^{37}$ kg m^2. The factors $1 + A_{\mathrm{f}}/A_{\mathrm{m}}$ are therefore 1.1264 for PREM and 1.1242 for CGGM. The dynamic ellipticity of the CMB is about 2.555×10^{-3} for PREM, 2.525×10^{-3} for the Wang model and 2.5824×10^{-3} for CGGM. The quantity β, as defined in Sasao *et al.* (1980), is 0.627×10^{-3} for the Wang model, according to Sasao & Wahr (1981). There are reasons to believe

Table 2. Reciprocal values of the geometrical flattening f, the precession constant H and the dynamic ellipticity e. Values at the inner core boundary, the core–mantle boundary and the outer surface are denoted by subscripts i, c and s, respectively.

Earth model	PREM	CGGM	PREMM
f_{i}^{-1}	412.6	398.6	402.7
H_{i}^{-1}	413.1	399.1	403.2
e_{i}^{-1}	412.1	398.1	402.2
f_{c}^{-1}	392.2	385.9	386.0
H_{c}^{-1}	392.9	386.6	386.7
e_{c}^{-1}	391.9	385.6	385.7
f_{s}^{-1}	299.69	298.15	298.16
H_{s}^{-1}	308.5	306.6	306.6
e_{s}^{-1}	307.5	305.6	305.6

that the latter constitutes a maximum possible value. Thus, accepting for the moment this value, we obtain $T_{\mathrm{FCN}}=460.5$ sidereal days for PREM, and $T_{\mathrm{FCN}}=454.9$ for CGGM.

Without describing in detail the complicated formulation of Sasao *et al.* (1980; see also Moritz & Mueller 1988, pp. 168–179) which leads to eq. (168), we should like to stress that the physical interpretation of (168) is rather obvious. Indeed, the period of the FCN should be shorter the larger the inertia of the fluid core (A_{f}) relative to the inertia of the solid mantle (A_{m}) and the larger the flattening of the CMB. Moreover, in agreement with Le Châtelier's principle (e.g. Atkins 1995), which states that when a system at equilibrium is subjected to a disturbance, it responds in a way that tends to minimize the effect of the disturbance, the 'rigid' period $T^{\mathrm{o}}_{\mathrm{FCN}}=[(1+A_{\mathrm{f}}/A_{\mathrm{m}})e_{\mathrm{c}}]^{-1}$ should become lengthened by the deformation, hence the factor $e_{\mathrm{c}}-\beta$, with $\beta>0$ in (168), instead of the simple factor e_{c}. The 'rigid' period $T^{\mathrm{o}}_{\mathrm{FCN}}$ is 347.5 sidereal days for PREM and 344.5 sidereal days for CGGM. The difference, i.e. 3 days, should represent within a factor of about 2 the uncertainty in T_{FCN} which arises from our uncertainty in the value of the dynamical core ellipticity e_{c}. Plausible variations of e_{c} with respect to a nominal 'hydrostatic' value caused by non-hydrostatic effects cannot provide larger corrections, and thus are unable to make up for the discrepancy of 25–30 days.

Working backwards, we notice that in order to achieve a FCN period of 430 days, the β-value for PREM should be 0.490×10^{-3}. For CGGM we should have $\beta=0.514\times10^{-3}$. As far as we can judge from the paper of Sasao *et al.* (1980) and the subsequent papers on the FCN topic, the value of β was computed by means of a code using Saito's (1974) formulation of static deformation. According to what we have seen in Section 5, there is no solution to the static deformation equations, neither for PREM nor for CGGM, both models possessing a fluid core which is not in neutral equilibrium (Denis *et al.* 1997). Saito's solutions are not divergence-free in the core, and thus cannot represent a situation of static equilibrium, because with these solutions isopycnics, isobars and level surfaces do not coincide in the deformed core.

A correct calculation of β remains to be performed, but the essential reason for the discrepancy between the observed and the computed period of FCN should be that indicated here. There seems to be no need in a first-order approximation to consider non-hydrostatic structure such as detailed CMB topography, which is difficult to determine by seismological means, nor more or less far-fetched effects such as anelasticity, viscomagnetic core–mantle coupling, inner core effects, and others. Such physical processes are of utmost importance for the study of the detailed dynamics of the Earth's interior and problems related to the geodynamo, but we believe they are not essential for the study of global deformations and motions in space of the Earth as a whole.

8 FINAL DISCUSSION AND CONCLUSIONS

As stated in the Introduction, one of the motivations for writing this paper was the recent paper by Abad *et al.* (1995), which convinced us once more that the use of variational methods may not be the *nec plus ultra* for obtaining accurate values of the oblateness of internal layers in the Earth or other cosmic bodies. A thorough review of useful methods for computing the Earth's figure functions therefore seemed appropriate. As far as our understanding of variational or virial techniques reaches, we believe that they are an elegant and powerful tool for solving some problems of existence of equilibrium figures, or establishing the stability or not of such equilibrium structures, but they are seldomly reliable enough when accurate numerical results are needed. Chandrasekhar's (1969) stimulating work provides a rather exhaustive treatment of the subject. Early attempts to apply virial techniques to an elastic Earth model were made by Grafarend & Hauer (1978) and by Hauer (1979). We think that altogether their results have not been too successful, and the method they used can definitely not compete with our straightforward method explained in Section 4.

Thus, having gained much experience with different efficient methods to compute the flattening, our first motivation was to review (this was done in Sections 2 and 3) all the necessary concepts and formulae of the internal gravity field theory in such a way as to make them easily accessible to everyone interested. Instead of going from the general theory to particular approximations, we decided, for reference purposes, to give first a rather complete account of three important approaches to deriving accurate results in the first-order (Clairaut) approximation. The latter embodies essentially the level of precision needed in geodynamical modelling. The results presented here concerning the general solutions in the inner core (Section 3.1.2) are published for the first time. For readers interested in more precise results, we then give a sufficiently detailed account of the second-order internal gravity field theory, again through three different approaches: an integro-differential approach, which we believe to be the most efficient in this case, as well as two differential approaches, namely the procedures of Darwin and de Sitter. Finally, we provide a general review of third-order theory, on which our numeric code CLAIRAUT3 is based. This program, written in Pascal, is available on request from the first author. The explanations are detailed enough for a general acquaintance with third- and higher-order theory, but a more complete review can be found in Denis (1989). In Section 4 we indicate a completely different approach to computing the shape of a planetary body, which leads to essentially the same results as first-order CLL theory. It is based on the elastic deformation equations, and thus may be applied to fluid as well as solid spheres. We indicate two different approaches. Although the results in both cases are quite similar, we want to stress that the assumption of an incompressible material is *not* trivial, whereas the consideration of incompressive flows is. Notice that Chambat (1992) also tried to compute non-hydrostatic equilibrium figures, but took a different approach.

Computing static figures of equilibrium by means of the global deformation equations gave us a second motivation for writing this paper, namely to discuss several unsolved geodynamic problems. The first of these problems we deal with concerns static deformations and the Longman paradox. We demonstrate mathematically that no static solutions, in particular no static Love numbers, exist if the model comprises a fluid core which is not neutrally stratified, as well as a solid mantle. We indicate that much of the debate about the static core dilemma arose because of erroneous assumptions or misleading arguments, some of which we discuss in detail. The fact that static solutions sometimes do not exist in situations where common sense would indicate they should exist is nicely illustrated by von Zeipel's theorem. Computing static solutions, in particular static tidal or load Love numbers, involves necessarily an assumption which is meaningless from a physical point of view. Generally, some physically unrealistic constraint is assumed on the core–mantle boundary. In this way, we come up with marred values of parameters that may be critical for some theory. A striking example is provided by the discrepancy of about 30 days between the observed and the computed period of the free core nutation. The latter is essentially due to the fact that deformation is computed with the quasi-static equations, which neglect the inertial acceleration term.

Incorrect arguments involving the concept of a global static deformation occur in many papers, despite the fact that Love had stated as early as 1911 that a static solution is impossible for an Earth model of uniform density made up of a solid mantle and a liquid core.

Next we show briefly how the internal gravity field theory may be extended to cope with a particular type of differential rotation. Many papers in geodynamics are marred by the fact that their authors consider a centrifugal potential when there is differential rotation of the fluid core with respect to the mantle, or of the inner core with respect to the mantle or outer core. In fact, centrifugal force derives from a potential only in the case of rigid body rotation or in the case of differential rotation which occurs on cylindrical shells having a common axis. In particular, the CLL theory, which relies essentially on the concept of level surfaces, cannot be used to deal with situations where the outer core is spinning with respect to the mantle or the inner core.

We then proceed to an overview of hydrostatic theory in relation to the Earth's density structure, and claim that the assumption of hydrostatic equilibrium pervades all our Earth models in a way which is very difficult to assess precisely. We give arguments (which we feel to be compelling) why the Earth's hydrostatic flattening f should not be defined by means of the Clairaut–Radau theory in terms of the mean inertia coefficient y (*cf.* eq. 159), but rather by means of the fundamental Clairaut formula (*cf.* eq. 162). The reason is that y is not a directly measurable quantity except if we *assume* that the mean inertia coefficient is very close to the polar inertia coefficient J_2/H. On the contrary, Clairaut's equation defines f in terms of two quantities, J_2 and m, which are measured with great accuracy. In this way, we obtain a hydrostatic value $f^{-1} \approx 298.6$, which is in much better agreement with the observed value 298.257 than the value 299.9 which is traditionally considered to represent *the* reciprocal hydrostatic flattening. Our definition has the further advantage of providing a hydrostatic value for the precessional constant H much closer to the observed value than formerly. In this way we derive a hydrostatic mean inertia coefficient of 0.3320, instead of the now commonly accepted value 0.3306. Thus, Earth models based on the constraint $y = 0.332$ yield theoretical values of f, J_2 and H which are in better agreement with the observed values than previously. Of course, perfect agreement cannot be achieved because the Earth is definitely not in hydrostatic balance. An important result of our way of looking at things is that with our models, which consistently use hydrostatic theory, we can remove, or at least alleviate, a number of discrepancies between theoretical and observed parameters which, until now, had represented rather difficult challenges.

We did not deal here with the interesting topic of evaluating palaeokinetic parameters in relation to a study of the tidal evolution of the Earth–Moon system and tectonics. The latter has already been considered by e.g. Denis (1986), Denis & Varga (1990) and Varga & Denis (1990). Some recent investigations are reported in a paper by Varga, Denis & Varga (1998).

ACKNOWLEDGMENTS

We are grateful to the authorities responsible for the Projet Tournesol, namely the Ministère des Affaires Etrangères in Paris and the Communauté Française de Belgique in Brussels, for financial support. CD also acknowledges with gratitude the travel grants awarded to him by the FNRS (Brussels). We are pleased that this work has recently led to the start of a collaboration with a scientist from the Institute of Geophysics of the University of Mining and Metallurgy in Kraków, Poland. Several referees had the difficult task of reviewing the manuscript of this long paper. We believe that some of their criticisms were unfounded, but some others were quite judicious and suggested improvements. We acknowledge their comments and help with pleasure and gratitude.

CD remembers with some melancholy that his first steps in the theory of equilibrium figures were guided by a series of lectures given at the Institut d'Astrophysique de Liège, much too long ago, by Prof. S. Chandrasekhar, and later by his students and co-workers, N. R. Lebovitz and M. Aizenman. He wishes to dedicate this small work to the memory of Prof. S. Chandrasekhar.

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